Higher (U, M)- Derivations in Completely Semi Prime Γ-Rings

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The aim of the present paper is to prove $d_n(u\alpha m) = \sum_{i+j=n} d_i(u)\alpha d_j(m), \forall u \in U, m \in M, \alpha \in \Gamma, n \in N$, where M is a 2-torsion free completely semiprime Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a,b,c \in M$ and $\alpha,\beta \in \Gamma$, U is an admissible Lie ideal of M and $D = (d_i)_{i\in N}$ is a higher (U,M)-derivation of M.

Keywords: Admissible Lie ideal, higher (U, M)-derivation, completely semiprime Γ-ring.

2010 AMS Subject Classification: Primary 13N15; Secondary 16W10,17C50.

1. INTRODUCTION

If there is a mapping $M \times \Gamma \times M \to M$ (M and Γ are additive abelian groups) such that

(i) $(x + y)\alpha z = x\alpha z + y\alpha z,\ x(\alpha + \beta)y = x\alpha y + x\beta y,\ x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z), \forall\ x, y, z \in M;\ \alpha, \beta \in \Gamma$,

then M is called a *Γ-ring*. This concept is more general than that of a ring. Barnes [1] generalized the notion of this type of ring which was introduced by Nobusawa [2]. A Γ-ring M is a *prime Γ-ring* if $\forall a, b \in M, a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$, M is called *semiprime* if $a\Gamma M\Gamma a = 0$ (with $a \in M$) implies $a = 0$ and it will be completely *semiprime* if $a\Gamma a = 0$ (with $a \in M$) implies $a = 0$. A Γ-ring M is *2-torsion free* if $2a = 0$ implies $a = 0, \forall a \in M$.

For any $x, y \in M$ and $\alpha \in \Gamma$, we induce a new product, the *Lie product* by $[x, y]_\alpha = x\alpha y - y\alpha x$. An additive subgroup $U \subset M$ is said to be a *Lie ideal* of M if whenever $u \in U, m \in M$ and $\alpha \in \Gamma$, then $[u, m]_\alpha \in U$. In the main results of this article we assume that the Lie ideal U verifies $u\alpha u \in U, \forall u \in U$. A Lie ideal of this type is called a *square closed Lie ideal*. Furthermore, if the Lie ideal U is square closed and U is not contained in $Z(M)$, where $Z(M)$ denotes the center of M, then U is called an *admissible Lie ideal* of M.

Throughout the article, we use the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$ and this is represented by (*). We make the basic commutator identities:

$[x\alpha y, z]_\beta = [x, z]_\beta\alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta$, $[x, y\alpha z]_\beta = [x, y]_\beta\alpha z - y[\alpha, \beta]_x z + y\alpha[x, z]_\beta$, $\forall a, b, c \in M, \forall\alpha, \beta \in \Gamma$. According to the condition (*), the above two identities reduces to

$[x\alpha y,z]_\beta = [x,z]_\beta \alpha y + x\alpha[y,z]_\beta, [x,y\alpha z]_\beta = [x,y]_\beta \alpha z + y\alpha[x,z]_\beta, \forall a,b,c \in M, \forall \alpha,\beta \in \Gamma$.

Herstein [3] proved a well known result in prime rings which states that 'every Jordan derivation on a 2-torsion free prime ring is a derivation'. Afterwards many mathematicians studied extensively the derivations in prime rings. (U,R)-derivations in rings have been introduced by Faraj, Haetinger and Majeed [4], as a generalization of Jordan derivations on a Lie ideal of a ring. The notion of (U,R)-derivation extends the concept given in [5]. Faraj, Haetinger and Majeed [4] proved that if R is a prime ring, char$(R) \neq 2$, U is a square closed Lie ideal of R and d is a (U,R)- derivation of R, then $d(ur) = d(u)r + ud(r), \forall, u \in U, r \in R$.

The notions of derivation and Jordan derivation in Γ-rings have been introduced by Sapanci and Nakajima [6]. Afterwards, in the light of some significant results due to Jordan left derivation of a classical ring obtained by Jun and Kim [7], some extensive results of left derivation and Jordan left derivation of a Γ-ring were determined by Ceven [8]. Halder and Paul [9] extended the results of Lie ideals [8].

In this article, we prove that $d_n(u\alpha m) = \sum_{i+j=n} d_i(u)\alpha d_j(m), \forall u \in U, m \in M, \alpha \in \Gamma, n \in N$, where M is a 2-torsion free completely semiprime Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a,b,c \in M$ and $\alpha,\beta \in \Gamma$, U is an admissible Lie ideal of M and $D = (d_i)_{i\in N}$ is a higher (U,M)-derivation of M.

2. (U,M)-DERIVATION

In this section we recall some definitions and lemmas (established earlier) that are important for representing our main objective in the later section.

Rahman and Paul [10] introduced the concept of (U,M) −derivation of Γ-ring in the following way:

Definition 1: Let M be a Γ-ring and U be a Lie ideal of M. An additive mapping $\mathrm{d}: \mathrm{M} \to \mathrm{M}$ is said to be a (U,M)- derivation of M if $\mathrm{d}(\mathrm{u}\alpha\mathrm{m} + \mathrm{s}\alpha\mathrm{u}) = \mathrm{d}(\mathrm{u})\alpha\mathrm{m} + \mathrm{u}\alpha\mathrm{d}(\mathrm{m}) + \mathrm{d}(\mathrm{s})\alpha\mathrm{u} + \mathrm{s}\alpha\mathrm{d}(\mathrm{u}), \forall \mathrm{u} \in \mathrm{U}, \mathrm{m}, \mathrm{s} \in \mathrm{M}$ and $\alpha \in \Gamma$.

Lemma 2.1: ([10, Lemma 1]) Let M be a 2-torsion free Γ-ring satisfying the condition (*), and U be a Lie ideal of M and d be a (U,M)-derivation of M, then

(i) $d(u\alpha m\beta u) = d(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u), \forall u \in U, m \in M$ and $\alpha,\beta \in \Gamma$.

(ii) $d(u\alpha m\beta v + v\alpha m\beta u) = d(u)\alpha m\beta v + u\alpha d(m)\beta v + u\alpha m\beta d(v) + d(v)\alpha m\beta u + v\alpha d(m)\beta u + v\alpha m\beta d(u), \forall u,v \in U, m \in M$ and $\alpha,\beta \in \Gamma$.

Definition 2: Let d be a (U,M)-derivation of M. We define $\phi_\alpha(\mathrm{u},\mathrm{m}) = \mathrm{d}(\mathrm{u}\alpha\mathrm{m}) - \mathrm{d}(\mathrm{u})\alpha\mathrm{m} - \mathrm{u}\alpha\mathrm{d}(\mathrm{m}), \forall \mathrm{u} \in \mathrm{U}, \mathrm{m} \in \mathrm{M}$ and $\alpha \in \Gamma$.

Lemma 2.2: Let d be a (U,M)-derivation of M, then $\forall \mathrm{u},\mathrm{v} \in \mathrm{U}, \mathrm{m} \in \mathrm{M}$ and $\alpha \in \Gamma$:

(i) $\phi_\alpha(u,m) = -\phi_\alpha(m,u)$

(ii) $\phi_\alpha(u+v,m) = \phi_\alpha(u,m) + \phi_\alpha(v,m), \forall u,v \in U, m \in M$ and $\alpha \in \Gamma$.

Lemma 2.3: ([10, Lemma 4]) Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U be a Lie ideal of M such that $U \not\subseteq Z(M)$. Then there exist elements $a, b \in U$ such that $[a,b]_\alpha = a\alpha b - b\alpha a \neq 0$.

Lemma 2.4: ([10, Lemma 5]) Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U be an admissible Lie ideal of M. If $t\alpha v\beta v + v\alpha v\beta t = 0$ for any $t \in M, v \in U$ and $\alpha, \beta \in \Gamma$, then $t = 0$.

Lemma 2.5: ([10, Theorem 1]) Let M be a 2-torsion free prime Γ-ring satisfying the condition (*), U be an admissible Lie ideal of M and d be a (U,M)-derivation of M. Then $\phi_\alpha(u,v) = 0, \forall u, v \in U$ and $\alpha \in \Gamma$.

Lemma 2.6: ([10, Lemma 6]) Let M be a 2-torsion free prime Γ-ring satisfying the condition (*), U be an admissible Lie ideal of M and d be a (U,M)-derivation of M. Then $\phi_\beta(u\alpha u, m) = 0, \forall u \in U, m \in M$ and $\alpha, \beta \in \Gamma$.

Theorem 2.1: ([10, Theorem 2]) Let M be a 2-torsion free prime Γ-ring satisfying the condition (*), U be a square closed Lie ideal of M and d be a (U,M)-derivation of M. Then $d(u\alpha m) = d(u)\alpha m + u\alpha d(m), \forall u \in U, m \in M, \alpha \in \Gamma$.

Definition 3. Let M be a Γ-ring and U be a Lie ideal of M and $D = (d_i)_{i \in N_0}$ be a family of additive mappings of M into itself such that $d_0 = id_M$, where id_M is an identity mapping on M. Then D is said to be a higher (U,M)-derivation of M if for each $n \in N, d_n(u\alpha m + s\alpha u) = \sum_{i+j=n} d_i(u)\alpha d_j(m) + d_i(s)\alpha d_j(u), \forall u \in U, m, s \in M$ and $\alpha, \beta \in \Gamma$.

Lemma 2.7: Let M be a 2-torsion free Γ-ring satisfying the condition (*), U be a Lie ideal of M and $D = (d_i)_{i \in N}$ be a higher (U,M)-derivation of M. Then $d_n(u\alpha m\beta u) = \sum_{i+j+k=n} d_i(u)\alpha d_j(m)\beta d_k(u), \forall u \in U, m \in M$ and $\alpha, \beta \in \Gamma$.

Proof: Let $x = u\alpha((2u)\beta m + m\beta(2u)) + ((2u)\beta m + m\beta(2u))\alpha u$.

We have, by Definition 3

$$d_n(u\alpha m + s\alpha u) = \sum_{i+j=n} d_i(u)\alpha d_j(m) + d_i(s)\alpha d_j(u) \qquad \text{(A)}$$

If $s = m$, then $d_n(u\alpha m + m\alpha u) = \sum_{i+j=n} d_i(u)\alpha d_j(m) + d_i(m)\alpha d_j(u)$ (B)

Now replacing m and s by $(2u)\beta m + m\beta(2u)$ and $(2u)\alpha m + m\alpha(2u)$ respectively in (A) and using (B) and the condition (*), we get

$$d_n(u\alpha((2u)\beta m + m\beta(2u)) + ((2u)\alpha m + m\alpha(2u))\beta u)$$
$$= \sum_{i+j=n} d_i(u)\alpha d_j((2u)\beta m + m\beta(2u)) + d_i((2u)\beta m + m\beta(2u))\alpha d_j(u)$$

This implies that

$$d_n(x) = 2\sum_{i+j=n} d_i(u)\alpha\sum_{l+t=j}\left(d_l(u)\beta d_t(m) + d_l(m)\beta d_t(u)\right)$$
$$+ 2\sum_{p+q=i}\sum_{i+j=n} d_j(u)\alpha\left(d_p(u)\beta d_q(m) + d_p(m)\beta d_q(u)\right)$$
$$= 2\sum_{i+l+t=n}\left(d_i(u)\alpha d_l(u)\beta d_t(m) + d_i(u)\alpha d_l(m)\beta d_t(u)\right)$$
$$+ 2\sum_{p+q+j=n}\left(d_p(u)\beta d_q(m)\alpha d_j(u) + d_p(m)\beta d_q(u)\alpha d_j(u)\right)$$
$$= 2\sum_{i+l+t=n}\left(d_i(u)\alpha d_l(u)\beta d_t(m) + d_i(u)\alpha d_l(m)\beta d_t(u)\right) +$$
$$2\textstyle\sum_{p+q+j=n}\left(d_p(u)\alpha d_q(m)\beta d_j(u) + d_p(m)\alpha d_q(u)\beta d_j(u)\right) \quad (1)$$

On the other hand by the definition of higher (U, M)-derivation and using the condition (*)

$$d_n(x) = 2d_n\left((u\alpha u)\beta m + m\beta(u\alpha u)\right) + 2d_n(u\alpha m\beta u) + 2d_n(u\beta m\alpha u)$$
$$= 2d_n((u\alpha u)\beta m + m\beta(u\alpha u)) + 2d_n(u\alpha m\beta u) + 2d_n(u\alpha m\beta u))$$
$$= 2\sum_{i+j=n}\left(d_i(u\alpha u)\beta d_j(m) + d_i(m)\beta d_j(u\alpha u)\right) + 4d_n(u\alpha m\beta u)$$
$$= 2\sum_{i+j=n}\sum_{r+s=i} d_r(u)\alpha d_s(u)\beta d_j(m) + 2\sum_{i+j=n}\sum_{e+k=j} d_i(m)\alpha d_e(u)\beta d_k(u)$$
$$+ 4d_n(u\alpha m\beta u)$$
$$= 2\textstyle\sum_{r+s+j=n} d_r(u)\alpha d_s(u)\beta d_j(m) + 2\sum_{i+e+k=n} d_i(m)\beta d_e(u)\alpha d_k(u) + 4d_n(u\alpha m\beta u) \quad (2)$$

Now comparing (1) and (2) we get

$4d_n(u\alpha m\beta u) = 4\sum_{i+j+k=n} d_i(u)\alpha d_j(m)\beta d_k(u), \forall u \in U, m \in M$ and $\alpha, \beta \in \Gamma$.

Using 2-torsion freeness of M we get the desired result.

Lemma 2.8: Let M be a 2-torsion free Γ-ring satisfying the condition (*), U be a Lie ideal of M and $D = (d_i)_{i\in N}$ be a higher (U, M)-derivation of M. Then $d_n(u\alpha m\beta v + v\alpha m\beta u) = \sum_{i+j+k=n} d_i(u)\alpha d_j(m)\beta d_k(v) + d_i(v)\alpha d_j(m)\beta d_k(u), \forall u, v \in U, m \in M$ and $\alpha, \beta \in \Gamma$.

Proof: Linearizing of $d_n(u\alpha m\beta u) = \sum_{i+j+k=n} d_i(u)\alpha d_j(m)\beta d_k(u)$ with respect to u gives

$$d_n\left((u+v)\alpha m\beta(u+v)\right) = \sum_{i+j+k=n} d_i(u+v)\alpha d_j(m)\beta d_k(u+v)$$
$$= \sum_{i+j+k=n}(d_i(u)\alpha d_j(m)\beta d_k(u) + d_i(u)\alpha d_j(m)\beta d_k(v)$$
$$+ d_i(v)\alpha d_j(m)\beta d_k(u) + d_i(v)\alpha d_j(m)\beta d_k(v)) \quad (3)$$

On the other hand

$$d_n\left((u+v)\alpha m\beta(u+v)\right) = d_n(u\alpha m\beta u) + d_n(u\alpha m\beta v + v\alpha m\beta u) + d_n(v\alpha m\beta v)$$
$$= \sum_{i+j+k=n}(d_i(u)\alpha d_j(m)\beta d_k(u) + d_n(u\alpha m\beta v + v\alpha m\beta u)$$
$$+ \textstyle\sum_{i+j+k=n}(d_i(v)\alpha d_j(m)\beta d_k(v) \quad (4)$$

Now comparing (3) and (4) we get

$d_n(u\alpha m\beta v + v\alpha m\beta u) = \sum_{i+j+k=n} d_i(u)\alpha d_j(m)\beta d_k(v) + d_i(v)\alpha d_j(m)\beta d_k(u), \forall u, v \in U, m \in M$ and $\alpha, \beta \in \Gamma$.

Definition 4: Let M be a 2-torsion free Γ-ring satisfying the condition (*) and U be a Lie ideal of M. For every higher (U, M)-derivation $D = (d_i)_{i\in N}$ of M, we define

$\phi_n^\alpha(u, m) = d_n(u\alpha m) - \sum_{i+j=n} d_i(u)\alpha d_j(m), \forall u \in U, m \in M, \alpha \in \Gamma$ and $n \in N$.

Lemma 2.9: Let M be a 2-torsion free Γ-ring satisfying the condition (*) and U be a Lie ideal of M. For every $u \in U, m \in M, \alpha \in \Gamma$ and $n \in N$, then $\phi_n^\alpha(u, m) + \phi_n^\alpha(m, u) = 0$.

Lemma 2.10: Let M be a 2-torsion free Γ-ring satisfying the condition (*) and let $D = (d_i)_{i\in N}$ be a higher (U, M)-derivation of M. Let $n \in N$ and assume that $a, b \in U, \alpha, \beta, \gamma \in \Gamma$. If $\phi_p^\alpha(a, b) = 0$, for every $p < n$, then $\phi_n^\alpha(a, b) = 0, \forall a, b \in U, \alpha \in \Gamma$ and $n \in N$.

Proof: Let $G = d_n(a\alpha b\beta[a, b]_\alpha\gamma b\alpha a + b\alpha a\beta[a, b]_\alpha\gamma a\alpha b)$.

First we compute $G = d_n(a\alpha(b\beta[a, b]_\alpha\gamma b)\alpha a) + d_n(b\alpha(a\beta[a, b]_\alpha\gamma a)\alpha b)$.

Using Lemma 2.7, we have

$$\begin{aligned} G &= \sum_{i+f+l=n} d_i(a)\alpha d_f(b\beta[a, b]_\alpha\gamma b)\alpha d_l(a) + \sum_{i+f+l=n} d_i(b)\alpha d_f(a\beta[a, b]_\alpha\gamma a)\alpha d_l(b) \\ &= \sum_{i+j+k+h+l=n} d_i(a)\alpha d_j(b)\beta d_k([a, b]_\alpha)\gamma d_h(b)\alpha d_l(a) \\ &\quad + \sum_{i+j+k+h+l=n} d_i(b)\alpha d_j(a)\beta d_k([a, b]_\alpha)\gamma d_h(a)\alpha d_l(b) \end{aligned}$$

On the other hand

$$G = d_n((a\alpha b)\beta[a, b]_\alpha\gamma(b\alpha a) + (b\alpha a)\beta[a, b]_\alpha\gamma(a\alpha b)).$$

Using Lemma 2.8, we obtain

$$G = \sum_{r+s+t=n} d_r(a\alpha b)\beta d_s([a, b]_\alpha)\gamma d_t(b\alpha a) + \sum_{r+s+t=n} d_r(b\alpha a)\beta d_s([a, b]_\alpha)\gamma d_t(a\alpha b).$$

Comparing both expression of G, we obtain

$\sum_{i+j+k+h+l=n} d_i(a)\alpha d_j(b)\beta d_k([a, b]_\alpha)\gamma d_h(b)\alpha d_l(a) +$
$\sum_{i+j+k+h+l=n} d_i(b)\alpha d_j(a)\beta d_k([a, b]_\alpha)\gamma d_h(a)\alpha d_l(b) =$
$\sum_{r+s+t=n} d_r(a\alpha b)\beta d_s([a, b]_\alpha)\gamma d_t(b\alpha a) + \sum_{r+s+t=n} d_r(b\alpha a)\beta d_s([a, b]_\alpha)\gamma d_t(a\alpha b)$

This implies

$\sum_{i+j+k+h+l=n} d_i(a)\alpha d_j(b)\beta d_k([a,b]_\alpha)\gamma d_h(b)\alpha d_l(a) -$
$\sum_{r+s+t=n} d_r(a\alpha b)\beta d_s([a,b]_\alpha)\gamma d_t(b\alpha a) =$
$-[\sum_{i+j+k+h+l=n} d_i(b)\alpha d_j(a)\beta d_k([a,b]_\alpha)\gamma d_h(a)\alpha d_l(b) -$
$\sum_{r+s+t=n} d_r(b\alpha a)\beta d_s([a,b]_\alpha)\gamma d_t(a\alpha b)]$ (5)

By the inductive assumption we can put $d_r(x\alpha y)$ for $\sum_{i+j=r} d_i(x)\alpha d_j(y)$, when $r < n$, for $x = a, b$ and $y = b, a$.

Thus, we can write by Definition 4

$$\sum_{i+j+k+h+l=n}(d_i(a)\alpha d_j(b)\beta d_k([a,b]_\alpha)\gamma d_h(b)\alpha d_l(a)) - \sum_{r+s+t=n}(d_r(a\alpha b)\beta d_s([a,b]_\alpha)\gamma d_t(b\alpha a)) = -(\phi_n^\alpha(a,b)\beta[a,b]_\alpha\gamma b\alpha a + a\alpha b\beta[a,b]_\alpha\gamma\phi_n^\alpha(b,a)) \quad (6)$$

Now using (6), (5) becomes

$\phi_n^\alpha(a,b)\beta[a,b]_\alpha\gamma b\alpha a + a\alpha b\beta[a,b]_\alpha\gamma\phi_n^\alpha(b,a) + \phi_n^\alpha(b,a)\beta[a,b]_\alpha\gamma a\alpha b +$
$b\alpha a\beta[a,b]_\alpha\gamma\phi_n^\alpha(a,b) = 0.$

By Lemma 2.4, we have

$$\phi_n^\alpha(a,b)\beta[a,b]_\alpha\gamma b\alpha a - a\alpha b\beta[a,b]_\alpha\gamma\phi_n^\alpha(a,b) - \phi_n^\alpha(a,b)\beta[a,b]_\alpha\gamma a\alpha b + b\alpha a\beta[a,b]_\alpha\gamma\phi_n^\alpha(a,b) = 0.$$

This implies that

$\phi_n^\alpha(a,b)\beta[a,b]_\alpha\gamma[a,b]_\alpha + [a,b]_\alpha\beta[a,b]_\alpha\gamma\phi_n^\alpha(a,b) = 0.$

Thus by Lemma 2.6 we get $\phi_n^\alpha(a,b) = 0, \forall a,b \in U, \alpha \in \Gamma$ and $n \in N$.

Lemma 2.11: Let M be a 2-torsion free prime Γ-ring satisfying the condition (*), U be an admissible Lie ideal of M and $\mathrm{D} = (\mathrm{d_i})_{\mathrm{i}\in\mathrm{N}}$ be a higher (U,M)-derivation of M. Then $\phi_\mathrm{n}^\alpha(\mathrm{u}\beta\mathrm{u},\mathrm{m}) = 0, \forall \mathrm{u} \in \mathrm{U}, \mathrm{m} \in \mathrm{M}, \alpha, \beta \in \Gamma$ and $\mathrm{n} \in \mathrm{N}$.

Proof: By Lemma 2.10, we have $\phi_n^\alpha(u,v) = 0, \forall u,v \in U$ and $\alpha \in \Gamma$.

Replacing v by $u\beta m - m\beta u$, we get

$0 = \phi_n^\alpha(u, u\beta m - m\beta u) = d_n(u\alpha u\beta m) - d_n(u\alpha m\beta u) - \sum_{i+j=n} d_i(u)\alpha d_j(u\beta m - m\beta u) =$
$d_n(u\alpha u\beta m) - d_n(u\alpha m\beta u) - \sum_{i+j=n} (d_i(u)\alpha \sum_{p+q=j} d_p(u)\beta d_q(m) - d_p(m)\beta d_q(u)) =$
$d_n(u\alpha u\beta m) - d_n(u\alpha m\beta u) - \sum_{i+p+q=n} d_i(u)\alpha d_p(u)\beta d_q(m) +$
$\sum_{i+p+q=n} d_i(u)\alpha d_p(m)\beta d_q(u) = d_n(u\alpha u\beta m) - \sum_{s+q=n} (\sum_{i+p=s} d_i(u)\alpha d_p(u))\beta d_q(m) =$
$\phi_n^\alpha(u\beta u, m).$

3. (U,M)-DERIVATION IN COMPLETELY SEMIPRIME Γ-RING

Now we prove our main result in the following way:

Theorem 3.1: Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*), U be an admissible Lie ideal of M and $\mathrm{D}=(\mathrm{d_i})_{\mathrm{i}\in\mathrm{N}}$ be a higher (U,M) -derivation of M. Then $\mathrm{d_n}(\mathrm{u}\alpha\mathrm{m})=\sum_{\mathrm{i+j=n}}\mathrm{d_i}(\mathrm{u})\alpha\mathrm{d_j}(\mathrm{m}),\forall\mathrm{u}\in\mathrm{U},\mathrm{m}\in\mathrm{M},\alpha\in\Gamma$ and $\mathrm{n}\in\mathrm{N}$.

Proof: By the definition $\phi_0^\alpha(u,m)=0,\forall u\in U,m\in M,\alpha\in\Gamma$.

Also by Theorem 2.1, $\phi_1^\alpha(u,m)=0,\forall u\in U,m\in M,\alpha\in\Gamma$.

Now we proceed by induction as

Suppose that $\phi_p^\alpha(u,m)=0,\forall u\in U,m\in M,\alpha\in\Gamma$ and $p\in N$

This implies that

$d_p(u\alpha m)=\sum_{i+j=p}d_i(u)\alpha d_j(m),u\in U,m\in M$ and $\alpha\in\Gamma$ and $p<n$, where $p,n\in N$

Since $D=(d_i)_{i\in N}$ is a higher (U,M)-derivation of M, thus we have

$$d_n(u\alpha(u\beta m)+(u\beta m)\alpha u)=\sum_{i+j=n}d_i(u)\alpha d_j(u\beta m)+d_i(u\beta m)\alpha d_j(u)=u\alpha d_n(u\beta m)+d_n(u)\alpha(u\beta m)+\sum_{i+j=n}^{i,j<n}d_i(u)\alpha d_j(u\beta m)+(u\beta m)\alpha d_n(u)+d_n(u\beta m)\alpha(u)+\sum_{i+j=n}^{i,j<n}d_i(u\beta m)\alpha d_j(u)=$$
$$u\alpha d_n(u\beta m)+d_n(u)\alpha(u\beta m)+\sum_{i+j=n}^{i,j<n}d_i(u)\alpha\sum_{s+t=j}d_s(u)\beta d_t(m)+(u\beta m)\alpha d_n(u)+d_n(u\beta m)\alpha(u)+\sum_{i+j=n}^{i,j<n}\sum_{l+q=i}d_l(u)\beta d_q(m)\alpha d_j(u)=u\alpha d_n(u\beta m)+d_n(u)\alpha(u\beta m)+\sum_{i+s+t=n}^{i,s+t<n}d_i(u)\alpha d_s(u)\beta d_t(m)+(u\beta m)\alpha d_n(u)+d_n(u\beta m)\alpha(u)+\sum_{l+q+j=n}^{l+q,j<n}d_l(u)\beta d_q(m)\alpha d_j(u) \tag{7}$$

On the other hand, using Lemma 2.7 and 2.11, we get

$$d_n(u\alpha(u\beta m)+(u\beta m)\alpha u)=d_n(u\alpha u\beta m)+d_n(u\beta m\alpha u)=\sum_{i+j=n}d_i(u\alpha u)\beta d_j(m)+\sum_{i+j+k=n}d_i(u)\beta d_j(m)\alpha d_k(u)=$$
$$\sum_{p+q+j=n}d_p(u)\alpha d_q(u)\beta d_j(m)+\sum_{i+j+k=n}d_i(u)\beta d_j(m)\alpha d_k(u)=d_n(u)\alpha u\beta m+u\alpha\sum_{q+j=n}d_q(u)\beta d_j(m)+\sum_{p+q+j=n}^{p,q+j<n}d_p(u)\alpha d_q(u)\beta d_j(m)+u\beta m\alpha d_n(u)+\sum_{i+j=n}d_i(u)\beta d_j(m)\alpha u+\sum_{i+j+k=n}^{i+j,k<n}d_i(u)\beta d_j(u)\alpha d_k(m) \tag{8}$$

By comparing (7) and (8)and using the condition (*), we get

$$\phi_n^\alpha(u,m)\beta u+u\beta\phi_n^\alpha(u,m)=0,\forall u\in U,m\in M,\alpha,\beta\in\Gamma \text{ and } n\in N \tag{9}$$

Linearizing of (9) with respect to u, gives us

$\phi_n^{\alpha}(u,m)\beta v + \phi_n^{\alpha}(v,m)\beta u + u\beta\phi_n^{\alpha}(v,m) + v\beta\phi_n^{\alpha}(u,m) = 0, \forall u, v \in U, m \in M, \alpha, \beta \in \Gamma$ and $n \in N$

Replacing v by $v\alpha v$ and using Lemma 2.4, we get $\phi_n^{\alpha}(u,m) = 0, \forall u \in U, m \in M, \alpha \in \Gamma$ and $n \in N$. Consequently, we get $d_n(u\alpha m) = \sum_{i+j=n} d_i(u)\alpha d_j(m), \forall u \in U, m \in M, \alpha \in \Gamma$ and $n \in N$.

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