Euler's Type Integrals Results for Matrix Argument involving Zonal Polynomial

Dr. P.K. Mishra[1], Dr. Abhijit Kulshreshtha[2,*] and Dr. Vaibhav Jain[3]
[1]Department of Mathematics, Sobhasaria Group of Institutions, Sikar, Rajasthan India.
[2,*]Department of Physics, JIET Group of Institutions, Jodhpur, Rajasthan, India.
*E-mail:- abhijitjecrc@yahoo.com
[3]HOD, Department of Physics, DAV (PG) College, Bulandshahr, UP, India.

In this paper some Euler's type integrals of generalized hypergeometric function of symmetric positive definite matrix associated with zonal polynomial are discussed, then established new result in particular cases and compare with Euler's type integral of scalar case. The integrals of Euler's type have a wide range of applications in the theory of applicable mathematics including statistical distributions. Few known and new results are discussed as particular cases.

2010 Mathematics Subject Classifications: 33C20, 33C45, 33E20, 33C60.

Keywords: Euler's type integral, Matrix arguments, Zonal polynomials, Hypergeometric function.

1. INTRODUCTION

Many writers discussed the theory of special function of matrix argument. Recently Mathai and Saxena [1] discussed the various property of special function of matrix argument. In this paper some integrals associated with matrix arguments due to Constantine [2] and James [3,4,5] have been evaluated with the help of the results given by Constantine [2] and Subrahmanian [6]. A number of integrals have been deduced from these theorems. These integral formulae are applicable to various multivariate distribution theory. It is also found useful in solving problems based on integral equations. The present study is in a way an extension in the line of approach initiated by Constantine [2], Mathai and Saxena [1] and Subrahmanian [6], R.J. Muirhead [7].

1.1. Notation

All the matrices are assumed pxp symmetric positive definite. Unless otherwise stated S, T, R > 0 means S, T and R in positive definite. Re(.) means that real part of (.), (.)ds, means that (.) in integrated out over the p × p symmetric matrix S. Such that S>0, I - S > 0 that in all the eigen values of S and are between 0 and 1. The notation k (.), k stands from norms of (.). Here since the matrices are symmetric and positive definite the largest eigen values can be taken as the norms. | (.) | stands for determination of (.), tr (.) denotes the trace of (.) that is the sum of leading diagonal elements of (.).

1.2. Hyper Geometric Function

Generalised Hyper geometric function of matrix argument is given by, A.M. Mathai [8],

$$_{m}F_{n}(a_1,........,a_m;b_1,........,b_n;X)=\sum_{k=0}^{\infty}\sum_{K}\frac{(a_1)_K.........(a_m)_K}{(b_1)_K.........(b_n)_K}\frac{C_K(X)}{k!} \qquad (1.2.1)$$

Such that none of the denominator factors is equal to zero.

where $$(a)_K=\prod_{i=1}^{p}\left[a-\frac{i-1}{2}\right]_{k_i} \qquad (1.2.2)$$

$K=(k_1,........k_p)$, K = $k_1 \geq k_2 \geq k_3 \geq ....k_p \geq 0$

$$(a)_K=\frac{\Gamma_p(a,K)}{\Gamma_p(a)} \ \& \ (a)_K=a(a+1)........(a+k-1),\ (a)_0=1 \qquad (1.2.3)$$

$$\Gamma_p(a,K)=\pi^{m(m-1)/4}\prod_{i=1}^{p}\Gamma\left[a+k_i-\frac{1}{2}(i-1)\right] \qquad (1.2.4)$$

$$\Gamma_p(a)=\pi^{m(m-1)/4}\prod_{i=1}^{p}\Gamma\left[a-\frac{1}{2}(i-1)\right] \qquad (1.2.5)$$

And X is a real positive definite matrix of order pxp, $C_K(X)$ are the zonal polynomials.

1.3. Zonal Polynomial

Zonal polynomial associated with a matrix X and certain description of various type of polynomial and the development of the theory is available from Mathai [5], Subrahmanian [9], Poonam and P.L. Sethi [10]. Consider a pxp real positive definite matrix X. Let V_k be the vector space of homogenous polynomial g(X) of degree k in the p(p+1)/2 different elements of pxp symmetric matrix X. Consider a congruent transformation X→LXL by a non singular pxp real matrix L. A subspace $V_s \subset V_k$ is called invariant if $V_s \subset V_k$ for all nonsingular matrices L. If V_s has no proper invariant subspace, it is called irreducible invariant subspace. It can be shown that V_k decomposes into a direct sum of irreducible subspace V_k corresponding to each partitions K = $(k_1,k_2,k_3........k_p)$, K = $k_1 \geq k_2 \geq k_3 \geq ....k_p \geq 0$ into not more than p parts. Each subspace contains a unique one dimensional subspace invariant under the orthogonal group of linear transformations. These subspaces are generated by the zonal polynomials $U_k(X)$ which when normalised in a certain fashion give a zonal polynomial $C_K(X)$ defined by Subrahmanian [6], explicit forms of these polynomials are available for small value of k. For large value of k it will be extremely difficult to compute these polynomial. For handling elementary special functions of matrix argument, we need a few properties of these zonal polynomials. These properties will be sufficient to establish the result. One basic result which is an immediate consequence of the definition itself is that when X is 1x1 matrix, namely a scalar quantity

$$C_K(X) = x^k \tag{1.3.1}$$

Hence one can look upon $C_K(X)$ as a generalization of x^k. The exponential function has the following expansion, by Subrahmanian [6],

$$e^{tr(X)} = \sum_{k=0}^{\infty} \frac{[tr(X)]^k}{k!} = \sum_{k=0}^{\infty} \sum_K \frac{C_K(X)}{k!} \tag{1.3.2}$$

The binomial expansion is the following for (I-X)>0 that is, X= X' > 0 and all the eigen values of X are between 0 and 1.

$$[I - X]^{-\alpha} = \sum_{k=0}^{\infty} \sum_K \frac{(\alpha)_k C_K(X)}{k!} \tag{1.3.3}$$

$$\int_{U>0} C_K(H' XHT) dU = \frac{C_K(X) C_K(T)}{C_K(I)} \tag{1.3.4}$$

2. RESULTS REQUIRED IN SEQUAL

Type I Beta Integration by Mathai [8], equation (12.3.10)

$$\int_{U>0} |U|^{a-(p+1)/2} |I - U|^{b-(p+1)/2} C_K(UX) dU = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b,K)} C_K(X) \tag{2.1}$$

where $\mathrm{Re}(a) > (p-1)/2$, $\mathrm{Re}(b) > (p-1)/2$

and $$\Gamma_p(a,K) = \pi^{p(p-1)/4} \prod_{j=1}^{p} \left(a + k_j - \frac{j-1}{2}\right) \tag{2.2}$$

3. MAIN RESULT

Consider a Integral of Hypergeometric function of matrix arguments

$$\int_{U>0} |U|^{a-(p+1)/2} |I - U|^{b-(p+1)/2} {}_mF_n(a_1,........,a_m;b_1,........,b_n;UX) dU$$

$$= \sum_{k=0}^{\infty} \sum_K \int_{U>0} |U|^{a-(p+1)/2} |I - U|^{b-(p+1)/2} \frac{(a_1)_K ....(a_m)_K}{(b_1)_K ....(b_n)_K} \frac{C_K(UX)}{k!} dU \tag{3.1}$$

Using equation (2.1), we get

$$=\sum_{k=0}^{\infty}\sum_{K}\frac{\Gamma_p(a,K)\Gamma_p(b)}{\Gamma_p(a+b,K)}\frac{(a_1)_K\,..........(a_m)_K}{(b_1)_K\,..........(b_n)_K}\frac{C_K(X)}{k!}$$

$$=\frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}\sum_{k=0}^{\infty}\sum_{K}\frac{(a)_K}{(a+b)_K}\frac{(a_1)_K\,..........(a_m)_K}{(b_1)_K\,..........(b_n)_K}\frac{C_K(X)}{k!}$$

$$=\frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}\,{}_{m+1}F_{n+1}(a,a_1,a_2......a_m;a+b,b_1,b_2......b_n;X) \tag{3.2}$$

4. METHODS AND DISCUSSIONS

Consider Euler's type integral by Harold Exton [11], (A.1.1.2)

$$\int_0^1 \mathrm{u}^{a-1}(1-\mathrm{u})^{b-1}\,{}_0F_1(-;c;ux)du=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,{}_1F_2(a;a+b,c;x) \tag{4.1}$$

Now if we take m=0, n=1 and b_1=c in (3.1), we get

$$\int_{U>0}|\mathrm{U}|^{a-(p+1)/2}\,|I-U|^{b-(p+1)/2}\,{}_0F_1(-;c;UX)dU=\frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}\,{}_1F_2(a;a+b,c;X) \tag{4.2}$$

which is matrix form of Euler's type's integration (4.1)

Now consider another Euler's type integral by Harold Exton [11], (A.1.1.9)

$$\int_0^1 \mathrm{u}^{a-1}(1-\mathrm{u})^{b-1}\,{}_1F_1(a+b;a;ux)du=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}e^x \tag{4.3}$$

Now if we take m=1, n=1 and a_1=a+b & b_1=a in (3.1), we get

$$\int_{U>0}|\mathrm{U}|^{a-(p+1)/2}\,|I-U|^{b-(p+1)/2}\,{}_1F_1(a+b;a;UX)dU=\frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}\sum_{k=0}^{\infty}\sum_{K}\frac{C_K(X)}{k!} \tag{4.4}$$

which is matrix form of Euler's type's integration (4.3)

Another Euler's type integral by Harold Exton [11], (A.1.1.10)

$$\int_0^1 \mathrm{u}^{a-1}(1-\mathrm{u})^{b-1}\,{}_1F_1(c;a;ux)du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,{}_1F_1(c;a+b;x) \qquad (4.5)$$

Now if we take m=1, n=1 and a_1=c & b_1=a in (3.1), we get

$$\int_{U>0} |\mathrm{U}|^{a-(p+1)/2}\,|I-U|^{b-(p+1)/2}\,{}_1F_1(c;a;UX)dU = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}\,{}_1F_1(c;a+b,X) \qquad (4.6)$$

which is matrix form of Euler's type's integration (4.5)

Another Euler's type integral by Harold Exton [11], (A.1.1.11)

$$\int_0^1 \mathrm{u}^{a-1}(1-\mathrm{u})^{b-1}\,{}_1F_1(c;d;ux)du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,{}_2F_2(a,c;a+b,d;x) \qquad (4.7)$$

Now if we take m=1, n=1 and a_1=c & b_1=d in (3.1), we get

$$\int_{U>0} |\mathrm{U}|^{a-(p+1)/2}\,|I-U|^{b-(p+1)/2}\,{}_1F_1(c;d;UX)dU = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}\,{}_2F_2(a,c;a+b,d;X) \qquad (4.8)$$

which is matrix form of Euler's type's integration (4.7)

Another Euler's type integral by Harold Exton [11], (A.1.1.40)

$$\int_0^1 \mathrm{u}^{c-1}(1-\mathrm{u})^{b-c-1}\,{}_2F_1(a,b;c;ux)du = \frac{\Gamma(c)\Gamma(b-c)}{\Gamma(b)}(1-x)^{-a} \qquad (4.9)$$

Now if we take a=c, b=b-c, m=2, n=1 and a_1=a, a_2=b & b_1=c in (3.1), we get

$$\int_{U>0} |\mathrm{U}|^{c-(p+1)/2}\,|I-U|^{b-c-(p+1)/2}\,{}_2F_1(a,b;c;UX)dU = \frac{\Gamma_p(c)\Gamma_p(b-c)}{\Gamma_p(b)}(I-X)^{-a} \qquad (4.10)$$

which is matrix form of Euler's type's integration (4.9)

Another Euler's type integral by Harold Exton [11], (A.1.1.33)

$$\int_0^1 \mathrm{u}^{c-1}(1-\mathrm{u})^{d-1}\,{}_2F_1(a,b;c;u)du = \frac{\Gamma(c)\Gamma(d)\Gamma(c+d-a-b)}{\Gamma(c+d-a)\Gamma(c+d-b)} \qquad (4.11)$$

Now if we take a=c, b=d, m=2, n=1 and a_1=a, a_2=b & b_1=c in (3.1), we get

$$\int_{U>0} |\mathrm{U}|^{c-(p+1)/2} |I-U|^{d-(p+1)/2} {}_2F_1(a,b;c;UX)dU = \frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(c+d)} \sum_{k=0}^{\infty}\sum_K \frac{(a)_k (b)_k}{(c+d)_k} \frac{C_K(X)}{k!}$$

For X = I unit matrix

$$\int_{U>0} |\mathrm{U}|^{c-(p+1)/2} |I-U|^{d-(p+1)/2} {}_2F_1(a,b;c;U)dU = \frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(c+d)} \sum_{k=0}^{\infty}\sum_K \frac{(a)_k (b)_k}{(c+d)_k} \frac{C_K(I)}{k!}$$

$$= \frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(c+d)} \frac{\Gamma_p(c+d)\Gamma_p(c+d-b-a)}{\Gamma_p(c+d-a)\Gamma_p(c+d-b)} = \frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(c+d-a)} \frac{\Gamma_p(c+d-b-a)}{\Gamma_p(c+d-b)} \quad (4.12)$$

which is matrix form of Euler's type's integration (4.11)

Another Euler's type integral by Harold Exton [11], (A.1.1.41)

$$\int_0^1 \mathrm{u}^{\mathrm{e}-1}(1-\mathrm{u})^{d-1} {}_2F_1(a,b;c;ux)du = \frac{\Gamma(e)\Gamma(d)}{\Gamma(e+d)} {}_3F_2(a,b,e;c,d+e;x) \quad (4.13)$$

Now if we take a=e, b=d, m=2, n=1 and a_1=a, a_2=b & b_1=c in (3.1), we get

$$\int_{U>0} |\mathrm{U}|^{e-(p+1)/2} |I-U|^{d-(p+1)/2} {}_2F_1(a,b;c;UX)dU = \frac{\Gamma_p(e)\Gamma_p(d)}{\Gamma_p(d+e)} {}_3F_2(a,b,e;c,d+e;X) \quad (4.14)$$

which is matrix form of Euler's type's integration (4.13)

Another Euler's type integral by Harold Exton [11], (A.1.1.80)

$$\int_0^1 \mathrm{u}^{\mathrm{a}-1}(1-\mathrm{u})^{b-1} {}_1F_2(a+b;c,a;ux)du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} {}_0F_1(-;c;x) \quad (4.15)$$

Now if we take m=1, n=2 and a_1= a+b & b_1=c, b_2=a in (3.1), we get

$$\int_{U>0} |\mathrm{U}|^{a-(p+1)/2} |I-U|^{b-(p+1)/2} {}_1F_2(a+b;a,c;UX)dU = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)} {}_0F_1(-;c;X) \quad (4.16)$$

which is matrix form of Euler's type's integration (4.15)

Another Euler's type integral by Harold Exton [11], (A.1.1.82)

$$\int_0^1 \mathrm{u}^{\mathrm{d-1}}(1-\mathrm{u})^{\,e-1}\,{}_1F_2(a;b,c;ux)du = \frac{\Gamma(d)\Gamma(e)}{\Gamma(d+e)}\,{}_2F_3(d,a;b,c,d+e;x) \tag{4.17}$$

Now if we take a=d, b=e, m=1, n=2 and a_1=a & b_1=b, b_2=c in (6.1), we get

$$\int_{U>0} |\mathrm{U}|^{d-(p+1)/2}\,|I-U|^{e-(p+1)/2}\,{}_1F_2(a,b;c;UX)dU = \frac{\Gamma_p(e)\Gamma_p(d)}{\Gamma_p(d+e)}\,{}_2F_3(a,d;b,c,d+e;X) \tag{4.18}$$

which is matrix form of Euler's type's integration (4.17)

Another Euler's type integral by Harold Exton [11], (A.1.1.112)

$$\int_0^1 \mathrm{u}^{\mathrm{e-1}}(1-\mathrm{u})^{\,c-e-1}\,{}_3F_2(a,b,c;d,e;ux)du = \frac{\Gamma(e)\Gamma(c-e)}{\Gamma(c)}\,{}_2F_1(a,b;d;x) \tag{4.19}$$

Now if we take a=e, b=c-e, m=3, n=2 and a_1=a, a_2=b, a_3=c & b_1=d, b_2=e in (3.1), we get

$$\int_{U>0} |\mathrm{U}|^{e-(p+1)/2}\,|I-U|^{c-e-(p+1)/2}\,{}_3F_2(a,b,c;d,e;UX)dU = \frac{\Gamma_p(e)\Gamma_p(c-e)}{\Gamma_p(c)}\,{}_2F_1(a,b;d;X) \tag{4.20}$$

which is matrix form of Euler's type's integration (4.19)

Another Euler's type integral by Harold Exton [11], (A.1.1.113)

$$\int_0^1 \mathrm{u}^{\mathrm{e-1}}(1-\mathrm{u})^{\,f-1}\,{}_3F_2(a,b,c;d,e;ux)du = \frac{\Gamma(e)\Gamma(f)}{\Gamma(e+f)}\,{}_3F_2(a,b,c;d,e+f;x) \tag{4.21}$$

Now if we take a=e, b=f, m=3, n=2 and a_1=a, a_2=b, a_3=c & b_1=d, b_2=e in (3.1), we get

$$\int_{U>0} |\mathrm{U}|^{e-(p+1)/2}\,|I-U|^{f-(p+1)/2}\,{}_3F_2(a,b,c;d,e;UX)dU = \frac{\Gamma_p(e)\Gamma_p(f)}{\Gamma_p(e+f)}\,{}_3F_2(a,b,c;d,e+f;X) \tag{4.22}$$

which is matrix form of Euler's type's integration (4.21)

Another Euler's type integral by Harold Exton [11], (A.1.1.114)

$$\int_0^1 \mathrm{u}^{\mathrm{f-1}}(1-\mathrm{u})^{\,g-1}\,{}_3F_2(a,b,c;d,e;ux)du = \frac{\Gamma(e)\Gamma(f)}{\Gamma(e+f)}\,{}_4F_3(a,b,c,f;d,e,f+g;x) \tag{4.23}$$

Now if we take a=f, b=g, m=3, n=2 and a_1=a, a_2=b, a_3=c & b_1=d, b_2=e in (3.1), we get

$$\int_{U>0} |U|^{f-(p+1)/2} |I-U|^{g-(p+1)/2} {}_3F_2(a,b,c;d,e;UX)dU = \frac{\Gamma_p(f)\Gamma_p(g)}{\Gamma_p(f+g)} {}_4F_3(a,b,c,f;d,e,f+g;X) \quad (4.24)$$

which is matrix form of Euler's type's integration (4.23)

Another Euler's type integral by Harold Exton [11], (A.1.1.115)

$$\int_0^1 u^{c-1}(1-u)^{d-1} {}_0F_3(-;a,b,c,;ux)du = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)} {}_0F_3(-;a,b,c+d;x) \quad (4.25)$$

Now if we take a=c, b=d, m=0, n=3 and b_1=a, b_2=b, b_3=c in (3.1), we get

$$\int_{U>0} |U|^{c-(p+1)/2} |I-U|^{d-(p+1)/2} {}_0F_3(-;a,b,c;UX)dU = \frac{\Gamma_p(c)\Gamma_p(d)}{\Gamma_p(c+d)} {}_0F_3(-;a,b,c+d;X) \quad (4.26)$$

which is matrix form of Euler's type's integration (4.25)

Another Euler's type integral by Harold Exton [11], (A.1.1.116)

$$\int_0^1 u^{d-1}(1-u)^{e-1} {}_0F_3(-;a,b,c;ux)du = \frac{\Gamma(d)\Gamma(e)}{\Gamma(d+e)} {}_1F_4(d;a,b,c,d+e;x) \quad (4.27)$$

Now if we take a=d, b=e, m=0, n=3 and b_1=a, b_2=b, b_3=c in (3.1), we get

$$\int_{U>0} |U|^{d-(p+1)/2} |I-U|^{e-(p+1)/2} {}_0F_3(-;a,b,c;UX)dU = \frac{\Gamma_p(d)\Gamma_p(e)}{\Gamma_p(d+e)} {}_1F_4(d;a,b,c,d+e;X) \quad (4.28)$$

which is matrix form of Euler's type's integration (4.27)

Another Euler's type integral by Harold Exton [11], (A.1.1.119)

$$\int_0^1 u^{e-1}(1-u)^{f-1} {}_2F_3(a,e+f;c,d.,e;ux)du = \frac{\Gamma(e)\Gamma(f)}{\Gamma(e+f)} {}_1F_2(a;c,d;x) \quad (4.29)$$

Now if we take a=e, b=f, m =2, n=3 and a_1=a, a_2=e+f; b_1=c, b_2=d, b_3=e in (3.1), we get

$$\int_{U>0} |U|^{e-(p+1)/2} |I-U|^{f-(p+1)/2} {}_2F_3(a,e+f;c,d,e;UX)dU = \frac{\Gamma_p(e)\Gamma_p(f)}{\Gamma_p(e+f)} {}_1F_2(a;c,d;X) \quad (4.30)$$

which is matrix form of Euler's type's integration (4.29)

Another Euler's type integral by Harold Exton [11], (A.1.1.120)

$$\int_0^1 u^{e-1}(1-u)^{f-1}\,{}_2F_3(a,b;c,d,e;ux)\,du = \frac{\Gamma(e)\Gamma(f)}{\Gamma(e+f)}\,{}_2F_3(a,b;c,d,e+f;x) \tag{4.31}$$

Now if we take a=e, b=f, m=2, n=3 and a_1=a, a_2=b; b_1=c, b_2=d, b_3=e in (3.1), we get

$$\int_{U>0} |U|^{e-(p+1)/2} |I-U|^{f-(p+1)/2}\,{}_2F_3(a,b;c,d,e;UX)\,dU = \frac{\Gamma_p(e)\Gamma_p(f)}{\Gamma_p(e+f)}\,{}_2F_3(a,b;c,d,e+f;X) \tag{4.32}$$

which is matrix form of Euler's type's integration (4.31)

Another Euler's type integral by Harold Exton [11], (A.1.1.121)

$$\int_0^1 u^{f-1}(1-u)^{g-1}\,{}_2F_3(a,b;c,d,e;ux)\,du = \frac{\Gamma(f)\Gamma(g)}{\Gamma(f+g)}\,{}_3F_4(a,b,f;c,d,e,f+g;x) \tag{4.33}$$

Now if we take a=f, b=g, m=2, n=3 and a_1=a, a_2=b; b_1=c, b_2=d, b_3=e in (3.1), we get

$$\int_{U>0} |U|^{f-(p+1)/2} |I-U|^{g-(p+1)/2}\,{}_2F_3(a,b;c,d,e;UX)\,dU = \frac{\Gamma_p(f)\Gamma_p(g)}{\Gamma_p(f+g)}\,{}_3F_4(a,b,f;c,d,e,f+g;X) \tag{4.34}$$

which is matrix form of Euler's type's integration (4.33)

Another Euler's type integral by Harold Exton [11], (A.1.1.134)

$$\int_0^1 u^{g-1}(1-u)^{d-g-1}\,{}_4F_3(a,b,c,d;e,f,g;ux)\,du = \frac{\Gamma(g)\Gamma(d-g)}{\Gamma(e+f)}\,{}_3F_2(a,b,c;e,f;x) \tag{4.35}$$

Now if we take a=g, b=d-g, m=4, n=3 and a_1=a, a_2=b, a_3=c, a_4=d; b_1=e, b_2=f, b_3=g in (3.1), we get

$$\int_{U>0} |U|^{g-(p+1)/2} |I-U|^{d-g-(p+1)/2}\,{}_4F_3(a,b,c,d;e,f,g;UX)\,dU = \frac{\Gamma_p(g)\Gamma_p(d-g)}{\Gamma_p(d)}\,{}_3F_2(a,b,c;e,f;X) \tag{4.36}$$

which is matrix form of Euler's type's integration (4.35)

Another Euler's type integral by Harold Exton [11], (A.1.1.135)

$$\int_0^1 u^{g-1}(1-u)^{h-1}\,{}_4F_3(a,b,c,d;e,f,g;ux)du = \frac{\Gamma(g)\Gamma(h)}{\Gamma(g+h)}\,{}_4F_3(a,b,c,d;e,f,g+h;x) \qquad (4.37)$$

Now if we take a=g, b=h, m=4, n=3 and a_1=a, a_2=b, a_3=c, a_4=d; b_1=e, b_2=f, b_3=g in (3.1), we get

$$\int_{U>0} |U|^{g-(p+1)/2} |I-U|^{h-(p+1)/2}\,{}_4F_3(a,b,c,d;e,f,g;UX)dU = \frac{\Gamma_p(g)\Gamma_p(h)}{\Gamma_p(g+h)}\,{}_4F_3(a,b,c,d;e,f,g+h;X) \qquad (4.38)$$

which is matrix form of Euler's type's integration (4.37)

Another Euler's type integral by Harold Exton [11], (A.1.1.136)

$$\int_0^1 u^{h-1}(1-u)^{j-1}\,{}_4F_3(a,b,c,d;e,f,g;ux)du = \frac{\Gamma(h)\Gamma(j)}{\Gamma(h+j)}\,{}_5F_4(a,b,c,d,h;e,f,g,j+h;x) \qquad (4.39)$$

Now if we take a=h, b=j, m=4, n=3 and a_1=a, a_2=b, a_3=c, a_4=d; b_1=e, b_2=f, b_3=g in (3.1), we get

$$\int_{U>0} |U|^{h-(p+1)/2} |I-U|^{j-(p+1)/2}\,{}_4F_3(a,b,c,d;e,f,g;UX)dU$$

$$= \frac{\Gamma_p(h)\Gamma_p(j)}{\Gamma_p(h+j)}\,{}_5F_4(a,b,c,d,h;e,f,g,j+h;X) \qquad (4.40)$$

which is matrix form of Euler's type's integration (4.39)

Acknowledgement

We are thankful to our management for their continuous encouragement to carry out this research work.

REFERENCE

[1] A.M. Mathai and R.K. Saxena; "Various practical problem in probability and statistics where Lauricella's confluent hypergeometric function appears naturally", Ganita Sandesh, Vol. 1(1-2), pp. 41-28, 1987.

[2] A.G. Constantine; "Some non-central distribution problem in multivariate analysis", Ann. Math. Statist., Vol. 34(4), pp. 1270-1285, 1963

[3] A.T. James; "The distribution of the latent roots of the covariance matrix", Ann. Math. Statist., Vol. 31(1), pp. 151-158,1960.

[4] A.T. James; "Distribution of matrix variates and latent roots derived from normal samples", Ann. Math. Statist., Vol. 35, pp. 475-501, 1964.

[5] A.T. James; "Inference of the latent root by calculation of hypergeometric function of matrix argument", Multivariates Analysis, (P.R. Krishnianh, eds.), Academic Press, New York, pp. 209-235, 1966.

[6] K. Subrahmanian; "Recent Trends in Multivariate Normal Distribution Theory: On the zonal polynomials and other functions of matrix argument", Sankhya, Vol. 38(3), pp. 221-258, 1976.

[7] R.J. Muirhead; "Expression for some hypergometric function of matrix argument with application", J. Multivariate Analysis, Vol. 5(3), pp. 283-293, 1975.

[8] A.M. Mathai; "Special Function and Function Of Matrix Argument: Recent developments And Recent Applications In Statistics And Astrophysics", Publication No. 32, 2005.

[9] K. Subrahmanian; "On some function of matrix argument", Utilitas Maths., Vol. 3(83), pp. 106, 1973.

[10] Poonam and P.L. Sethi; "Dual series involving Laguerre polynomials with matrix arguments",Indian Journal of Pure and Applied Maths., Vol. 23(1), pp. 49-53, 1992.

[11] Harold Exton; "Handbook of Hypergeometric Integrals: Theory, Applications, Tables, Computer programs", Ellis Horwood, pp. 152-171, 1978.