Generalized Hypergeometric Forms of Some Elliptic Type Integrals

Nadeem Ahmad
Department of Bio sciences, Jamia Millia Islamia, Jamia nagar, New Delhi, India.

The study of elliptic type integrals has been important due to their applications in certain problems involving computations of radiation field. In an arbitrary angular distribution law the radiation field off axis from a circular disc radiated. In this paper I have developed some generalized hypergeometric forms of elliptic type integrals in complete and incomplete integrals.

Keywords: elliptic type integrals, complete elliptic type integrals, incomplete elliptic type integrals.

1. INTRODUCTION

The generalized hypergeometric function of one variable [1, pp. 437] is defined as

$$ {}_AF_B\left[\begin{matrix} a_1, a_2, ..., a_A; \\ b_1, b_2, ..., b_B; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n ... (a_A)_n}{(b_1)_n (b_2)_n ... (b_B)_n} \frac{z^n}{n!} $$

OR
$$ {}_AF_B\left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{[(a_A)]_n}{[(b_B)]_n} \frac{z^n}{n!} \tag{1} $$

Where for the sake of convenience (in the contracted notation), (a_A) denotes the array of 'A' number of parameters given by $a_1, a_2, ..., a_A$. The denominator parameters are neither zero nor negative integers. The numerator parameters may be zero and negative integers. A and B are positive integers or zero. Empty sum is to be interpreted as zero and empty product as unity. $\sum_{n=a}^{b}$ and $\prod_{n=a}^{b}$ are empty if $b < a$.

Suppose that the numerator parameters are neither zero nor negative integers (otherwise the question of convergence will not arise).

If A=B+1, the series ${}_AF_B$ converges for |z|<1 and diverges for |z|>1

Wright's generalized hypergeometric function [2, pp. 50 (1.5.21), pp. 179 (34 iii), pp. 395 (23)], is defined by

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$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,A_1),...,(\alpha_p,A_p);\\(\beta_1,B_1),...,(\beta_q,B_q);\end{matrix}z\right]=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)..\Gamma(\alpha_p)}{\Gamma(\beta_1)\Gamma(\beta_2)..\Gamma(\beta_q)}{}_p\Psi_q^*\left[\begin{matrix}(\alpha_1,A_1),...,(\alpha_p,A_p);\\(\beta_1,B_1),...,(\beta_q,B_q);\end{matrix}z\right] \quad (2) $$

$$ =\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+A_1n)\Gamma(\alpha_2+A_2n)...\Gamma(\alpha_p+A_pn)}{\Gamma(\beta_1+B_1n)\Gamma(\beta_2+B_2n)...\Gamma(\beta_q+B_qn)}\frac{z^n}{n!} \quad (3) $$

$$ =H_{p,q+1}^{1,p}\left[-z\left|\begin{matrix}(1-\alpha_1,A_1),...,(1-\alpha_p,A_p);\\(0,1),(1-\beta_1,B_1),...,(1-\beta_q,B_q);\end{matrix}\right.\right] \quad (4) $$

$$ {}_p\Psi_q^*\left[\begin{matrix}(\alpha_1,A_1),...,(\alpha_p,A_p);\\(\beta_1,B_1),...,(\beta_q,B_q);\end{matrix}z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_{A_1n}(\alpha_2)_{A_2n}...(\alpha_p)_{A_pn}}{(\beta_1)_{B_1n}(\beta_2)_{B_2n}...(\beta_q)_{B_qn}}\frac{z^n}{n!} \quad (5) $$

Where the coefficients A_1, A_2, A_p and B_1, $B_{2,...,}$ B_q are positive real numbers.

The Fox H-function makes sense when either $\delta=(1+B_1+B_{2+...+}B_q)-(A_1+A_{2+...+}A_p)>0$ and $0<|z|<\infty$; $z\neq0$.

OR

The equality holds for suitably constrained value of |z|.

δ=0 and $0<|z|<R=A_1^{-A_1}A_2^{-A_2}...A_p^{-A_p}B_1^{-B_1}B_2^{-B_2}...B_q^{-B_q}$

$$ {}_p\Psi_q\left[\begin{matrix}(\alpha_1,1_1),...,(\alpha_p,1);\\(\beta_1,1_1),...,(\beta_q,1);\end{matrix}z\right]=\frac{\prod_{j=1}^{p}\Gamma(\alpha_j)}{\prod_{j=1}^{q}\Gamma(\beta_j)}{}_pF_q\left[\begin{matrix}\alpha_1,...,\alpha_p;\\\beta_1,...,\beta_q;\end{matrix}z\right] \quad (6) $$

If Re (M)>-1 and Re (N)>-1, then

$$ \int_0^{\pi/2}(\sin^M\theta\cos^N\theta)d\theta=\frac{\Gamma\left(\frac{M+1}{2}\right)\Gamma\left(\frac{N+1}{2}\right)}{2\Gamma\left(\frac{M+N+2}{2}\right)}, \quad (7) $$

$$ \sum_{p=0}^{\infty}A(p)=\sum_{p=0}^{\infty}A(2p)+\sum_{p=0}^{\infty}A(2p+1) \quad (8) $$

If $m=1,2,3,...$ and $n=0,1,2,3,...$ then

$$(b)_{mn} = m^{mn}\left(\frac{b}{m}\right)_n\left(\frac{b+1}{m}\right)_n \ldots \left(\frac{b+m-2}{m}\right)_n\left(\frac{b+m-1}{m}\right)_n \tag{9}$$

The notation denotes $\Delta(N;b)$ the array of *N* parameters given by $\frac{b}{N}, \frac{b+1}{N}, \ldots, \frac{b+n-1}{N}$. Where *N*=1, 2, 3,…

The incomplete elliptic integral of third kind [1] is defined by

$$\int_0^{\phi} \frac{d\theta}{(1+a^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \Pi(k,\phi,a) \tag{10}$$

Where $a \neq k, 0$, is called characteristic parameter of integral of third kind.

The complete elliptic integral of third kind [1] and its hypergeometric form [3, pp. 150 (25)] is given by

$$\int_0^{\pi/2} \frac{d\theta}{(1+a^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \Pi(k,a) = \frac{\pi}{2} F_1\left[\frac{1}{2};1,\frac{1}{2};1;-a^2,k^2\right] \tag{11}$$

$$(0 < k^2 < 1,\ -\infty < a^2 < +\infty,\ a^2 \neq -1)$$

Another incomplete elliptic integral [1, pp. 744] is given by

$$\int_0^{\phi} \frac{\sin^2\theta.d\theta}{\sqrt{1-k^2\sin^2\theta}} = D(k,\phi) \tag{12}$$

The complete elliptic integral [1, pp. 771-772] is given by

$$\int_0^{\pi/2} \frac{\sin^2\theta.d\theta}{\sqrt{1-k^2\sin^2\theta}} = D(k) = \frac{\pi}{4}\,{}_2F_1\left[\begin{matrix}\frac{3}{2},\frac{1}{2};\\ 2;\end{matrix}\,k^2\right] \tag{13}$$

In the book of L.C. Andrews [4, pp. 139 (Q.No. 13)] the following integral was represented in terms of K(k) and E(k) in the form

$$\int_0^{\pi} \frac{\cos\theta.d\theta}{\sqrt{a^2+b^2+c^2-2ab.\cos\theta}} = \frac{2}{k\sqrt{ab}}\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right] \tag{14}$$

Where $k^2 = \dfrac{4ab}{(a+b)^2 + c^2}$ and its hypergeometric form [3, pp. 151 (26)] is given by

$$\int_0^{\pi} \frac{\cos\theta.d\theta}{\sqrt{a^2+b^2+c^2-2ab.\cos\theta}} = \frac{\pi}{2}\frac{ab}{(a^2+b^2+c^2)^{3/2}}\ {}_2F_1\left[\begin{matrix}\frac{3}{4},\frac{5}{4};\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\\ 2;\end{matrix}\right] \qquad (15)$$

In the book of A. Erdélyi *et al.* [5, pp. 317-318, pp. 321-325] following elliptic integrals and their hypergeometric forms [3, pp. 151 (27, 28)] are given in the forms

$$\int_0^{\phi} \frac{\cos^2\theta.d\theta}{\sqrt{1-k^2\sin^2\theta}} = B(k,\phi) \qquad (16)$$

$$\int_0^{\pi/2} \frac{\cos^2\theta.d\theta}{\sqrt{1-k^2\sin^2\theta}} = B(k) = \frac{\pi}{4}\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2};k^2\\ 2;\end{matrix}\right] \qquad (17)$$

$$\int_0^{\phi} \frac{(\sin\theta\cos\theta)^2.d\theta}{\left(\sqrt{1-k^2\sin^2\theta}\right)^3} = C(k,\phi) \qquad (18)$$

$$\int_0^{\pi/2} \frac{(\sin\theta\cos\theta)^2.d\theta}{\left(\sqrt{1-k^2\sin^2\theta}\right)^3} = C(k) = \frac{\pi}{16}\,{}_2F_1\left[\begin{matrix}\frac{3}{2},\frac{3}{2};k^2\\ 3;\end{matrix}\right] \qquad (19)$$

The hypergeometric forms of complete elliptic integrals $K(k), E(k)$ and $D(k)$ are given in the literature on elliptic integrals.

To represent the above elliptic integrals, different notations were used [1, pp. 744, 771-772], [4, pp. 133-134], [6, pp. 224], [7, pp. 172-174], [8, pp. 174-181], [9, pp. 904-905] and [10].

2. MORE ELLIPTIC TYPE INTEGRAL AND THEIR HYPERGEOMETRIC FORMS

We can obtained the following hypergeometric forms of some elliptic type integrals

$$\int_0^{\pi/2} \frac{\sin^{2m}\theta.d\theta}{(a^2+b^2+c^2-2ab.\sin^{2n}\theta)^g} = \int_0^{\pi/2} \frac{\cos^{2m}\theta.d\theta}{(a^2+b^2+c^2-2ab.\cos^{2n}\theta)^g}$$

$$= \frac{\pi}{2} \frac{\left(\frac{1}{2}\right)_m}{(a^2+b^2+c^2)^g (1)_m} {}_2\Psi_1^* \left[\begin{matrix} (g,1), \left(\frac{1}{2}+m, n\right); \\ (1+m, n); \end{matrix} \left(\frac{2ab}{a^2+b^2+c^2} \right) \right] \tag{20}$$

When '*n*' is positive integer, we get the right hand side

$$= \frac{\pi}{2} \frac{\left(\frac{1}{2}\right)_m}{(a^2+b^2+c^2)^g (1)_m} {}_{n+1}F_n \left[\begin{matrix} g, \Delta\left(n; \frac{1}{2}+m\right); \\ \Delta(n; 1+m); \end{matrix} \left(\frac{2ab}{a^2+b^2+c^2} \right) \right] \tag{21}$$

2.1 Proof of expression (20) and (21)

Left hand side of expression (20) can be written as

$$L_1 = \frac{1}{(a^2+b^2+c^2)^g} \int_0^{\pi/2} \frac{\sin^{2m}\theta . d\theta}{\left[1 - \left(\frac{2ab}{a^2+b^2+c^2}\right)\sin^{2n}\theta\right]^g}$$

$$= \frac{1}{(a^2+b^2+c^2)^g} \int_0^{\pi/2} \sin^{2m}\theta \left[1 - \left(\frac{2ab}{a^2+b^2+c^2}\right)\sin^{2n}\theta\right]^{-g} d\theta$$

$$= \frac{1}{(a^2+b^2+c^2)^g} \sum_{k=0}^{\infty} \frac{(g)_k \left(\frac{2ab}{a^2+b^2+c^2}\right)^k}{k!} \int_0^{\pi/2} \sin^{2m+2nk}\theta . d\theta$$

$$= \frac{1}{(a^2+b^2+c^2)^g} \sum_{k=0}^{\infty} \frac{(g)_k \left(\frac{2ab}{a^2+b^2+c^2}\right)^k}{k!} \frac{\Gamma\left(\frac{2m+2nk+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2m+2nk+2}{2}\right)}$$

$$= \frac{1}{2(a^2+b^2+c^2)^g} \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(m+1)} \sum_{k=0}^{\infty} \frac{(g)_k \left(\frac{2ab}{a^2+b^2+c^2}\right)^k}{k!} \frac{\left(m+\frac{1}{2}\right)_{nk}}{(m+1)_{nk}}$$

$$= \frac{\pi\left(\frac{1}{2}\right)_m}{2(a^2+b^2+c^2)^g(1)_m}\sum_{k=0}^{\infty}\frac{(g)_k\left(m+\frac{1}{2}\right)_{nk}}{(m+1)_{nk}}\frac{\left(\frac{2ab}{a^2+b^2+c^2}\right)^k}{k!}$$

$$= \frac{\pi}{2}\frac{\left(\frac{1}{2}\right)_m}{(a^2+b^2+c^2)^g(1)_m}\,{}_2\Psi_1^*\left[\begin{matrix}(g,1),\left(\frac{1}{2}+m,n\right);\\(1+m,n);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right]$$

This is the right hand side of equation (20)

When '*n*' is positive integer and using (9), we get

$$L_1 = \frac{\pi}{2}\frac{\left(\frac{1}{2}\right)_m}{(a^2+b^2+c^2)^g(1)_m}\,{}_{n+1}F_n\left[\begin{matrix}g,\Delta\left(n;\frac{1}{2}+m\right);\\\Delta(n;1+m);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right]$$

This is the right hand side of equation (21)

$$\int_0^{\pi/2}\frac{\sin^{2m+1}\theta.d\theta}{(a^2+b^2+c^2-2ab.\sin^{2n+1}\theta)^g} = \int_0^{\pi/2}\frac{\cos^{2m+1}\theta.d\theta}{(a^2+b^2+c^2-2ab.\cos^{2n+1}\theta)^g}$$

$$= \frac{(1)_m}{(a^2+b^2+c^2)^g\left(\frac{3}{2}\right)_m}\,{}_2\Psi_1^*\left[\begin{matrix}(g,1),\left(1+m,\frac{2n+1}{2}\right);\\\left(m+\frac{3}{2},\frac{2n+1}{2}\right);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right] \quad (22)$$

When '*n*' is positive integer, we get the right hand side

$$= \frac{(1)_m}{(a^2+b^2+c^2)^g\left(\frac{3}{2}\right)_m}\,{}_{2n+3}F_{2n+2}\left[\begin{matrix}\Delta(2;g),,\Delta(2n+1;1+m);\\\frac{1}{2},\Delta(2n+1;\frac{3}{2}+m);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\right]$$

$$+\frac{\pi abg\left(\frac{3}{2}\right)_{m+n}}{2(a^2+b^2+c^2)^{g+1}(2)_{m+n}}\,{}_{2n+3}F_{2n+2}\left[\begin{matrix}\Delta(2;g+1),,\Delta\left(2n+1;\frac{3}{2}+m+n\right);\\ \frac{3}{2},\Delta(2n+1;2+m+n);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\right] \quad (23)$$

2.2 Proof of expression (22) and (23)

Left hand side of expression (22) can be written as

$$L_2=\frac{1}{(a^2+b^2+c^2)^g}\int_0^{\pi/2}\frac{\sin^{2m+1}\theta.d\theta}{\left[1-\left(\frac{2ab}{a^2+b^2+c^2}\right)\sin^{2n+1}\theta\right]^g}$$

$$=\frac{1}{(a^2+b^2+c^2)^g}\int_0^{\pi/2}\sin^{2m+1}\theta\left[1-\left(\frac{2ab}{a^2+b^2+c^2}\right)\sin^{2n+1}\theta\right]^{-g}d\theta$$

$$=\frac{1}{(a^2+b^2+c^2)^g}\sum_{k=0}^{\infty}\frac{(g)_k\left(\frac{2ab}{a^2+b^2+c^2}\right)^k}{k!}\int_0^{\pi/2}\sin^{2m+2nk+k+1}\theta.d\theta$$

$$=\frac{1}{(a^2+b^2+c^2)^g}\sum_{k=0}^{\infty}\frac{(g)_k\left(\frac{2ab}{a^2+b^2+c^2}\right)^k}{k!}\frac{\Gamma\left(\frac{2m+2+2nk+k}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2m+3+2nk+k}{2}\right)}$$

$$=\frac{1}{(a^2+b^2+c^2)^g}\sum_{k=0}^{\infty}\frac{(g)_k\Gamma\left(m+1+\left(\frac{2n+1}{2}\right)k\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(m+\frac{3}{2}+\left(\frac{2n+1}{2}\right)k\right)}\frac{\left(\frac{2ab}{a^2+b^2+c^2}\right)^k}{k!}$$

$$=\frac{1}{2(a^2+b^2+c^2)^g}\frac{\Gamma(m+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)}\sum_{k=0}^{\infty}\frac{(g)_k(m+1)_{\left(\frac{2n+1}{2}\right)k}}{\left(m+\frac{3}{2}\right)_{\left(\frac{2n+1}{2}\right)k}}\frac{\left(\frac{2ab}{a^2+b^2+c^2}\right)^k}{k!}$$

$$=\frac{(1)_m}{(a^2+b^2+c^2)^g\left(\frac{3}{2}\right)_m}\,{}_2\Psi_1^*\left[\begin{matrix}(g,1),\left(m+1,\frac{2n+1}{2}\right);\\ \left(m+\frac{3}{2},\frac{2n+1}{2}\right);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right]$$

This is the right hand side of equation (22)

When '*n*' is positive integer and applying the identity (8), we get

$$L_2=\frac{1}{(a^2+b^2+c^2)^g}\sum_{k=0}^{\infty}\frac{(g)_{2k}\left(\frac{2ab}{a^2+b^2+c^2}\right)^{2k}}{(2k)!}\int_0^{\pi/2}\sin^{2m+4nk+2k+1}\theta.d\theta$$

$$+\frac{1}{(a^2+b^2+c^2)^g}\sum_{k=0}^{\infty}\frac{(g)_{2k+1}\left(\frac{2ab}{a^2+b^2+c^2}\right)^{2k+1}}{(2k+1)!}\int_0^{\pi/2}\sin^{2m+4nk+2n+2k+2}\theta.d\theta$$

$$=\frac{1}{(a^2+b^2+c^2)^g}\sum_{k=0}^{\infty}\frac{\left(\frac{g}{2}\right)_k\left(\frac{g+1}{2}\right)_k\Gamma\left(\frac{2m+4nk+2k+2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)_k\left(\frac{2}{2}\right)_k 2\Gamma\left(\frac{2m+4nk+2k+3}{2}\right)}\left(\frac{2ab}{a^2+b^2+c^2}\right)^{2k}$$

$$+\frac{1}{(a^2+b^2+c^2)^g}\frac{(g)_1}{(1)_1}\left(\frac{2ab}{a^2+b^2+c^2}\right)$$

$$\times\sum_{k=0}^{\infty}\frac{\left(\frac{g+1}{2}\right)_k\left(\frac{g+2}{2}\right)_k\Gamma\left(\frac{2m+4nk+2n+2k+3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\left(\frac{2}{2}\right)_k\left(\frac{3}{2}\right)_k 2\Gamma\left(\frac{2m+4nk+2n+2k+4}{2}\right)}\left(\frac{2ab}{a^2+b^2+c^2}\right)^{2k}$$

$$=\frac{1}{2(a^2+b^2+c^2)^g}\frac{\Gamma(m+1)}{\Gamma\left(m+\frac{3}{2}\right)}\sum_{k=0}^{\infty}\frac{\left(\frac{g}{2}\right)_k\left(\frac{g+1}{2}\right)_k(m+1)_{(2n+1)k}}{\left(\frac{1}{2}\right)_k\left(m+\frac{3}{2}\right)_{(2n+1)k}}\frac{\left(\frac{2ab}{a^2+b^2+c^2}\right)^{2k}}{k!}$$

$$+\frac{abg}{(a^2+b^2+c^2)^{g+1}}\frac{\Gamma\left(m+n+\frac{3}{2}\right)}{\Gamma(m+n+2)}\sum_{k=0}^{\infty}\frac{\left(\frac{g+1}{2}\right)_k\left(\frac{g+2}{2}\right)_k\left(m+n+\frac{3}{2}\right)_{(2n+1)k}}{\left(\frac{3}{2}\right)_k(m+n+2)_{(2n+1)k}}\frac{\left(\frac{2ab}{a^2+b^2+c^2}\right)^{2k}}{k!}$$

$$=\frac{(1)_m}{(a^2+b^2+c^2)^g\left(\frac{3}{2}\right)_m}{}_{2n+3}F_{2n+2}\left[\begin{matrix}\Delta(2;g),,\Delta(2n+1;1+m);\\ \frac{1}{2},\Delta(2n+1;\frac{3}{2}+m);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\right]$$

$$+\frac{\pi abg\left(\frac{3}{2}\right)_{m+n}}{2(a^2+b^2+c^2)^{g+1}(2)_{m+n}}{}_{2n+3}F_{2n+2}\left[\begin{matrix}\Delta(2;g+1),,\Delta\left(2n+1;\frac{3}{2}+m+n\right);\\ \frac{3}{2},\Delta(2n+1;2+m+n);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\right]$$

This is the right hand side of equation (23)

Similarly, we can derive

$$\int_0^{\pi/2}\frac{\sin^{2m+1}\theta.d\theta}{(a^2+b^2+c^2-2ab.\sin^{2n}\theta)^g}=\int_0^{\pi/2}\frac{\cos^{2m+1}\theta.d\theta}{(a^2+b^2+c^2-2ab.\cos^{2n}\theta)^g}$$

$$=\frac{(1)_m}{(a^2+b^2+c^2)^g\left(\frac{3}{2}\right)_m}{}_2\Psi_1^*\left[\begin{matrix}(g,1),(m+1,n);\\ \left(m+\frac{3}{2},n\right);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right] \quad (24)$$

When '*n*' is positive integer, we get

$$=\frac{(1)_m}{(a^2+b^2+c^2)^g\left(\frac{3}{2}\right)_m}{}_{n+1}F_n\left[\begin{matrix}g,\Delta(n;1+m);\\ \Delta(n;\frac{3}{2}+m);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right] \quad (25)$$

$$\int_0^{\pi/2}\frac{\sin^{2m}\theta.d\theta}{(a^2+b^2+c^2-2ab.\sin^{2n+1}\theta)^g}=\int_0^{\pi/2}\frac{\cos^{2m}\theta.d\theta}{(a^2+b^2+c^2-2ab.\cos^{2n+1}\theta)^g}$$

$$=\frac{\pi\left(\frac{1}{2}\right)_m}{2(a^2+b^2+c^2)^g(1)_m}\,{}_2\Psi_1^*\left[\begin{matrix}(g,1),\left(m+\frac{1}{2},\frac{2n+1}{2}\right);\\ \left(m+1,\frac{2n+1}{2}\right);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right] \tag{26}$$

When '*n*' is positive integer, we get

$$=\frac{\pi\left(\frac{1}{2}\right)_m}{2(a^2+b^2+c^2)^g(1)_m}\,{}_{2n+3}F_{2n+2}\left[\begin{matrix}\Delta(2;g),,\Delta\left(2n+1;\frac{1}{2}+m\right);\\ \frac{1}{2},\Delta(2n+1;1+m);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\right]$$

$$+\frac{2abg(1)_{m+n}}{(a^2+b^2+c^2)^{g+1}\left(\frac{3}{2}\right)_{m+n}}\,{}_{2n+3}F_{2n+2}\left[\begin{matrix}\Delta(2;g+1),,\Delta(2n+1;1+m+n);\\ \frac{3}{2},\Delta(2n+1;\frac{3}{2}+m+n);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\right] \tag{27}$$

$$\int_0^{\pi/2}\frac{\sin^{2m}\theta.d\theta}{(a^2+b^2+c^2-2ab.\cos^{2n}\theta)^g}=\int_0^{\pi/2}\frac{\cos^{2m}\theta.d\theta}{(a^2+b^2+c^2-2ab.\sin^{2n}\theta)^g}$$

$$=\frac{\pi}{2}\frac{\left(\frac{1}{2}\right)_m}{(a^2+b^2+c^2)^g(1)_m}\,{}_2\Psi_1^*\left[\begin{matrix}(g,1),\left(\frac{1}{2},n\right);\\ (1+m,n);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right] \tag{28}$$

When '*n*' is positive integer, we get

$$=\frac{\pi}{2}\frac{\left(\frac{1}{2}\right)_m}{(a^2+b^2+c^2)^g(1)_m}\,{}_{n+1}F_n\left[\begin{matrix}g,\Delta\left(n;\frac{1}{2}\right);\\ \Delta(n;1+m);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right] \tag{29}$$

$$\int_0^{\pi/2}\frac{\sin^{2m+1}\theta.d\theta}{(a^2+b^2+c^2-2ab.\cos^{2n+1}\theta)^g}=\int_0^{\pi/2}\frac{\cos^{2m+1}\theta.d\theta}{(a^2+b^2+c^2-2ab.\sin^{2n+1}\theta)^g}$$

$$=\frac{(1)_m}{(a^2+b^2+c^2)^g\left(\frac{3}{2}\right)_m}\,{}_2\Psi_1^*\left[\begin{matrix}(g,1),\left(\frac{1}{2},\frac{2n+1}{2}\right);\\ \left(m+\frac{3}{2},\frac{2n+1}{2}\right);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right] \quad (30)$$

When '*n*' is positive integer, we get

$$=\frac{(1)_m}{(a^2+b^2+c^2)^g\left(\frac{3}{2}\right)_m}\,{}_{2n+3}F_{2n+2}\left[\begin{matrix}\Delta(2;g),,\Delta\left(2n+1;\frac{1}{2}\right);\\ \frac{1}{2},\Delta\left(2n+1;\frac{3}{2}+m\right);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\right]$$

$$+\frac{abg(1)_m(1)_n}{(a^2+b^2+c^2)^{g+1}(2)_{m+n}}\,{}_{2n+3}F_{2n+2}\left[\begin{matrix}\Delta(2;g+1),\Delta(2n+1;1+n);\\ \frac{3}{2},\Delta(2n+1;2+m+n);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\right] \quad (31)$$

$$\int_0^{\pi/2}\frac{\sin^{2m+1}\theta.d\theta}{(a^2+b^2+c^2-2ab.\cos^{2n}\theta)^g}=\int_0^{\pi/2}\frac{\cos^{2m+1}\theta.d\theta}{(a^2+b^2+c^2-2ab.\sin^{2n}\theta)^g}$$

$$=\frac{(1)_m}{(a^2+b^2+c^2)^g\left(\frac{3}{2}\right)_m}\,{}_2\Psi_1^*\left[\begin{matrix}(g,1),\left(\frac{1}{2},n\right);\\ \left(\frac{3}{2}+m,n\right);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right] \quad (32)$$

When '*n*' is positive integer, we get

$$=\frac{(1)_m}{(a^2+b^2+c^2)^g\left(\frac{3}{2}\right)_m}\,{}_{n+1}F_n\left[\begin{matrix}g,\Delta\left(n;\frac{1}{2}\right);\\ \Delta\left(n;\frac{3}{2}+m\right);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right] \quad (33)$$

$$\int_0^{\pi/2}\frac{\sin^{2m}\theta.d\theta}{(a^2+b^2+c^2-2ab.\cos^{2n+1}\theta)^g}=\int_0^{\pi/2}\frac{\cos^{2m}\theta.d\theta}{(a^2+b^2+c^2-2ab.\sin^{2n+1}\theta)^g}$$

$$=\frac{\pi\left(\frac{1}{2}\right)_m}{2(a^2+b^2+c^2)^g(1)_m}\,{}_2\Psi_1^*\left[\begin{matrix}(g,1),\left(\frac{1}{2},\frac{2n+1}{2}\right);\\ \left(m+1,\frac{2n+1}{2}\right);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)\right] \tag{34}$$

When '*n*' is positive integer, we get

$$=\frac{\pi\left(\frac{1}{2}\right)_m}{2(a^2+b^2+c^2)^g(1)_m}\,{}_{2n+3}F_{2n+2}\left[\begin{matrix}\Delta(2;g),,\Delta\left(2n+1;\frac{1}{2}\right);\\ \frac{1}{2},\Delta(2n+1;1+m);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\right]$$

$$+\frac{2abg\left(\frac{1}{2}\right)_m(1)_n}{(a^2+b^2+c^2)^{g+1}\left(\frac{3}{2}\right)_{m+n}}\,{}_{2n+3}F_{2n+2}\left[\begin{matrix}\Delta(2;g+1),\Delta(2n+1;1+n);\\ \frac{3}{2},\Delta\left(2n+1;\frac{3}{2}+m+n\right);\end{matrix}\left(\frac{2ab}{a^2+b^2+c^2}\right)^2\right] \tag{35}$$

Using the properties of definite integral, we can obtain the hypergeometric forms of elliptic integrals over the interval $[0,\pi]$ and $[0,2\pi]$.

3. DEDUCTIONS

- By setting g$=\frac{1}{2}, m=n=0$, and transforming the integral over the interval $[0,\pi]$ to the equation (23), then we can get the result given in equation (15).

- By putting $g=\frac{1}{2}, m=n=1$ and $k^2=\frac{2ab}{(a+b)^2+c^2}$ in equation (29), we can deduce the result (17).

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