Strongly NA – Continuous Mappings

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This paper introduces a new class of mappings, namely, strongly na-continuous mappings. This class lies properly between the class of super I-continuous mappings and the class of strongly continuous mappings due to Levine. Some characterizations and basic properties of these mappings are obtained. The relationships among these mappings and some other stronger forms of continuity are also investigated.

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1. PRELIMINARIES

For a topological space X and A⊂X, the θ- closure of A, denoted by θ-cl(A), is defined [10] to be the set of all x Є X, such that, every closed neighborhood of x intersects A and A⊂X is called θ-closed if A = θ-cl (A) . Similarly, θ-interior of A⊂X, denoted by θ-int (A), is defined to be the set of all x, for which, there exists a closed neighbourhood of x contained in A and A⊂X is called θ-open if A = θ-int (A). The complement of θ-open (resp. θ-closed) sets are θ-closed (resp. θ-open) sets. A⊂X is defined to be semi-open [3] if, there exists an open set O such that O ⊂ A ⊂ cl(O). Complement of a semi-open set is called a semi-closed set. Semi-closure of A, denoted by s-cl (A), is the smallest semi-closed super set of A. Semi-interior of A, denoted by s-int (A) is the largest semi-open set contained in A. Feebly open [6] set is defined as a set A such that O ⊂ A ⊂ s-cl (O) for an open set O.

2. DEFINITIONS AND CHARACTERIZATIONS

Definition 2.1: A mapping f : X → Y is said to be **strongly na-continuous** at x Є X if for each semi-open set B containing f (x), ∃ a θ-open set A containing x : f (A) ⊂ B. A mapping is said to be **strongly na- continuous** if it is strongly na-continuous at each x Є X.

Following theorem gives some characterizations for strongly na-continuous mappings.

Theorem 2.1: For a bijection f: X → Y, the following are equivalent:

(a) f is strongly na-continuous.

(b) f^{-1} (B) is θ-open for each semi open set B in Y,

(c) f^{-1} (B) is θ- closed for each semi closed set B in Y,

(d) f (θ-cl (A)) ⊂ s-cl (f (A)) for each subset A of X,

(e) θ-cl (f^{-1} (B)) ⊂ f^{-1} (s-cl (B)) for each subset B of Y,

(f) s- int (f (A)) ⊂ f (θ - int (A) for all subsets A of X.

(g) f^{-1} (s- int (B)) ⊂ θ- int (f^{-1} (B)) for all subsets B of Y.

Proof:

(a) => (b) Let F be any semi open set in Y and x Є f^{-1} (F), then f (x) Є F. By hypothesis, ∃ a θ- open set E containing x, such that, f (E) ⊂ F. Therefore, x Є E ⊂ f^{-1} (F). Now, E is θ - open and x Є E, therefore by cor. 1 [4] There exists a regular open set G, such that x Є G ⊂ cl (G) ⊂ E ⊂ f^{-1} (F). Thus, f^{-1} (F) is θ- open.

(b) => (c) Let F be a semi closed set in Y. Then Y – F is semi open and therefore by (b) f^{-1} (Y – F) = X – f^{-1} (F) is θ- open. Hence f^{-1} (F) is θ- closed.

(c) => (d) For each subset A of X, f (A) ⊂ s -cl (f (A)) => A ⊂ f^{-1} (s-cl (f (A)) => θ-cl (A) ⊂ θ-cl (f^{-1} (s-cl (f (A)))) => θ-cl(A) ⊂f^{-1}(s-cl(f(A))) ⇒ f (θ-cl (A)) ⊂ s-cl (f (A)).

(d) => (e) For each B ⊂ Y, f^{-1} (B) ⊂ X, so f (θ-cl (f^{-1} (B))) ⊂ s-cl (f (f^{-1} (B)) ⊂ s-cl (B) => θ-cl (f^{-1} (B)) ⊂ f^{-1} (s-cl (B)).

(e) => (a) Let x Є X, and f (x) Є V, where, V is any semi open set in Y. Then Y – V is semi closed, so, θ-cl (f^{-1} (Y – V) ⊂ f^{-1} (s - cl (Y – V)) = f^{-1} (Y – V) = X – f^{-1} (V).

Thus X – f^{-1} (V) is θ-closed. Putting f^{-1} (V) = U, we have f (U) ⊂ V.

(b) => (f) For a subset A of X, s- int (f (A)) ⊂ f (A) => f^{-1} (s- int (f (A))) ⊂ A => θ-int (f^{-1}(s - int (f (A)))) ⊂ θ -int A => f^{-1} (s-int (f (A))) ⊂ θ- int A, => s- int (f (A)) ⊂ f (θ- int A)).

(f) => (g) For each B ⊂ Y, f^{-1} (B) ⊂ X, therefore s- int (f (f^{-1} (B))) ⊂ f (θ- int (f^{-1} (B))) => s- int (B) ⊂ f (θ- int (f^{-1} (B))) => f^{-1} (s- int (B)) ⊂ θ- int (f^{-1} (B)).

(g) => (b) Let A be semi open in Y, then f^{-1} (s- int (A)) ⊂ θ- int (f^{-1} (A)) or, s- int (A) ⊂ f (θ- int (f^{-1} (A))) => A ⊂ f (θ- int (f^{-1} (A))) => f^{-1} (A) ⊂ θ- int (f^{-1} (A)) => f^{-1} (A) is θ-open.

3. BASIC PROPERTIES

Some of the basic properties of strongly na-continuous mappings are contained in

Theorem 3.1:

(a) Every constant mapping is strongly na-continuous.

(b) Every mapping from a discrete space is strongly na-continuous.

(c) Every mapping onto an indiscrete space is strongly na- continuous.

(d) Let f: X → (Y, T) be strongly na-continuous and T'⊂ T, then f: X → (Y, T') is strongly na- continuous.

(e) Composition of two strongly na- continuous mappings is strongly na- continuous mapping.

Theorem 3.2: Restriction of strongly na-continuous mapping to an open set is strongly na-continuous.

Proof: Let f: X → Y be strongly na- continuous, let A be an open set in X . Let V be semi open in Y, then f^{-1} (V) is θ–open in X. For x Є f / A^{-1}(V) there exists a regular open set U, such that, x Є U ⊂ cl (U)⊂ f^{-1} (V). Now U is a regular open and A is open, U ∩ A is regular open in the subspace A, and cl_A (U ∩ A) = cl_X (U ∩ A) ∩ A ⊂ cl_X(U) ∩ cl_X(A) ∩ A=cl_x(U) ∩ A ⊂ cl_X (U) ⊂ f^{-1}(V) so f/A^{-1}(V) is θ-open in A.

Theorem 3.3: The graph map of f: X → Y, defined as g(x) = (x, f (x)), is strongly na-continuous, implies f is strongly na-continuous.

Proof: Let x Є X and V be any semi open set in Y. Then X × V is semi is semi open in X × Y containing g(x) = (x, f(x)). Since, the graph map g : X → X × Y is strongly na-continuous, there exists a θ-open set U containing x, such that, g(U) ⊂ X × V. The implies that f(U) ⊂ V.

Theorem 3.4: A mapping f : X → Y is strongly na-continuous iff for each xϵX, and each semi open set V containing f(x), there exists an open set U containing x, : f(cl(U)) ⊂ V.

Proof: Let f : X → Y be strongly na-continuous, then for each xϵX, and each semi open set V containing f(x), ∃ a θ-open set B, : f(B)⊂ V. Now, B is θ-open, so, ∃ an open set U, such that, x Є U ⊂ cl(U) ⊂ B. Hence f(cl(U)) ⊂ V. Conversely, if to each semi open set V containing f(x), there exists an open set U, : f (cl(U)) ⊂ V, then cl(U) is itself θ-open.

Definition 3.1: [5] A space X is called **semi - T_0 - space** if to every pair of distinct points of X, ∃ a semi open set containing one but not the other.

Definition 3.2: [5] A space X is called **semi - T_2 - space** if to every pair of distinct points of X, ∃ disjoint semi open sets separating them.

Theorem 3.5: If f: X → Y is injective strongly na-continuous mapping and Y is semi T_0, then X is T_2.

Proof: If x, y are distinct points of X, and f is injective, then f(x) and f(y) are distinct points in Y. If Y, semi T_0, then, there exists semi open set V in Y containing, say, f(x) but not f(y). If f is strongly na-continuous, then ∃ an open set, U,: x∈U and f(cl(U)) ⊂ V. Now, f(y)∉V and so, y∉cl(U). Therefore, U and X–cl(U) are disjoint open sets in X containing x and y, respectively.

Corollary 3.1: If f: X → Y is strongly na-continuous injection and Y is semi T_2. Then X is Urysohn.

Theorem 3.6: If f and g are strongly na-continuous mappings from a space X into semi T_2 space Y. Then the set A = {x: f(x) = g(x)} is θ-closed in X.

Proof: Let y∉A, then f(y)≠g(y), then there exists disjoint semi open sets P and Q in Y. such that, g(y) Є P and f(y) Є Q. Since both the mappings are strongly na-continuous, so $g^{-1}(P)$ and $f^{-1}(Q)$ are θ-open in X. Therefore, E = $g^{-1}(P) \cap f^{-1}(Q)$ is also θ-open. Now, we claim E ⊂ X – A. Suppose, if possible, z ∉ X– A. Then z ∊ E ⇒ f(z)∊Q and g(z) ∊ P, z ∉ X – A ⇒ f(z) = g(z), but this contradicts P ∩ Q = ϕ.

Corollary 3.2: If f and g are super I-continuous mappings from a space X into a semi T_2 space Y. Then the set A = {x: f(x) = g(x)} is δ-closed.

4. COMPARISIONS

Definition 4.1: [2] A mapping f: X → Y is said to be **strongly continuous** if for every subset A of X, f(cl(A)) ⊂ f(A).

It is proved [2] that f: X → Y is strongly continuous if and only if the inverse image of every subset of Y is open (or closed). Obviously, every strongly continuous mapping is strongly na-continuous but the converse is not true. The identity map on an indiscrete space with at least two points, is strongly na-continuous but not strongly continuous.

Definition 4.2: [8] A mapping f : X → Y is **strongly θ-continuous** if to each x Є X, and each open set V containing f(x), ∃ an open set U containing x, : f(cl(U)) ⊂ V.

Obviously, every strongly na-continuous mapping is strongly θ-continuous but the converse is not true because the identity map on the real line (R, U) is strongly θ-continuous but not strongly na-continuous.

Definition 4.3: A mapping f : X → Y is **na-continuous** [1] **super continuous** [7] **super I-continuous** [11] if $f^{-1}(U)$ is δ-open in X for each feebly open, open, semi open set U respectively in Y.

Clearly, every strongly na-continuous mapping is super I-continuous but the converse is not true.

Example 4.1: [1] Let X = {a, b, c, d}, T = {ϕ, X, {c}, {a, b}, {a, b, c}} and Y = {x, y, z}, U = {ϕ, Y, {x, y}}, f : (X, T) → (Y, U) be defined as f(a) = x, f(b) = y, f(c) = f(d) = z. Then f is super I-continuous but not strongly na-continuous. Also, every super I-continuous mapping is na-continuous but the converse is not true.

Example 4.2: [1] Let X= {a, b, c}, T = {ϕ, X, {a}, {b}, {a, b}}. Let g: X → X be a mapping defined by g(a) = g(b)= a and g(c)= c. Then g is na-continuous but not super I-continuous.

Thus, we have, following implication diagram:

Strongly continuous mapping ⇒ Strongly na-continuous ⇒ super I-continuous mapping ⇒na-continuous mapping ⇒ Super continuous mapping ⇒ continuous mapping

But none of the implication is reversible.

Definition 4.4: [9] A space is defined to be **almost regular** if for each x∈X and each regular closed set A not containing x, ∃ disjoint open sets U and V : x∈ U and A⊂Y.

Theorem 4.1: Let f: X → Y be super I-continuous, and X almost regular space, then f is strongly na-continuous.

Proof: Let V be semi open in Y. Then $f^{-1}(V)$ is δ-open. To show $f^{-1}(V)$ is θ-open. To show $f^{-1}(V)$ θ-open, let x∈ $f^{-1}(V)$, then, ∃ a regular open set U, : x∈U⊂$f^{-1}(V)$. Now, x∈U by theorem 2.2 [9] ∃ a regular open set A, : x∈A ⊂ cl(A) ⊂ U. So, $f^{-1}(V)$ is θ-open.

Definition 4.5: [11] A mapping f: X → Y is said to be **I-continuous (resp. completely I-continuous, totally I-continuous)** if $f^{-1}(A)$ is open (resp. regular open, clopen) in X, for each open set A in Y.

Corollary 4.1: If f: X → Y is completely I-continuous and X is almost regular. Then f is strongly na-continuous.

Corollary 4.2: If X is regular and f: X → Y is I-continuous. Then f is strongly na-continuous.

Corollary 4.3: If f: X → Y is totally I-continuous and X is almost regular (or regular). Then f is strongly na-continuous.

Theorem 4.2:- Every na-continuous mapping f: X → Y from an almost regular space X into extremally disconnected space Y, is strongly na-continuous.

Proof: Let A be semi open in Y, then A ⊂ cl(int(A)). Y is extremally disconnected space, so, cl(int(A)) = int(cl(int(A))). Thus, A is α-open, or, A is feebly open. Now, f is na-continuous, so $f^{-1}(A)$ is δ-open. X is almost regular, so, $f^{-1}(A)$ is θ-open.

Corollary 4.4:- If X is almost regular and Y is extremally disconnected space, then for f: X → Y (a) f is strongly na-continuous (b)f is super I-continuous (c) f is na-continuous, are equivalent conditions.

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