ξ-Regular Spaces

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The aim of this paper is to introduce and study a new class of regular space called ξ-regular spaces by using ξ-open sets introduced by R. Devi, S.N. Rajappriya, K.M. Swamy and H. Maki in Scientiae Mathematical Japonicae (2006) and obtained several properties of such a space. Moreover, we obtained some new characterizations and preservation theorems of ξ-regular spaces.

Keywords: ξ, ξ*, ξ**, gα, rgα-closed sets, ξ, ξ*, ξ**, gα, rgα-open sets, ξ-open, Almost ξ-open, Pre ξ-closed, ξg-closed, Almost ξg-closed functions, ξ-regular spaces.

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1. INTRODUCTION

The aim of this paper is to introduce a new class of normal spaces called ξ-regular spaces by using ξ-open sets and obtained several properties of such a space. R. Devi *et al.* [1] introduced the concept of ξ, ξ*, ξ**-closed sets and discuss some of their basic properties. R. Devi *et al.* [2] introduced the concept of α- regular space by using α -open sets and obtained several properties of such a space. Throughout this paper (X,τ), (Y, σ) spaces always mean topological spaces X, Y respectively on which no separation axioms are assumed unless explicitly stated.

2. PRELIMINARIES

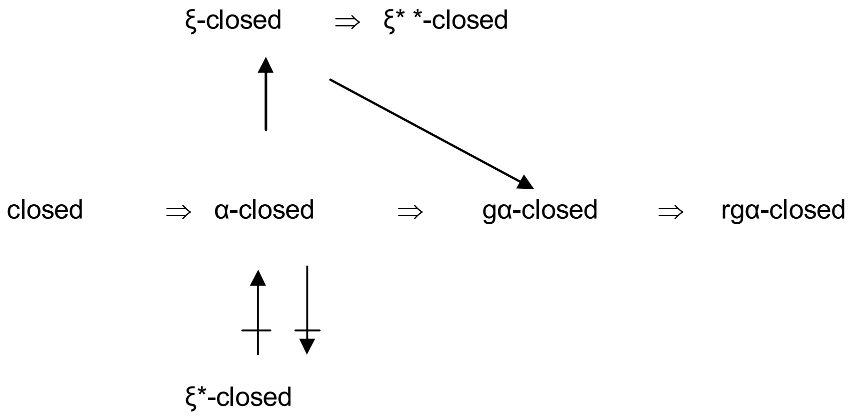
2.1. Definition 1: A subset of a space X is called

1. **Regular closed** [3] if A = cl(int(A)).
2. **α-closed** [4] if cl(int(cl(A))) $\subseteq$ A .
3. **gα-closed** [5] if α-cl(A) $\subseteq$ U whenever A $\subseteq$ U, and U is an α-open in X.
4. **ξ- closed** [1] if α-cl(A) $\subseteq$ U whenever A $\subseteq$ U, and U is a gα-open in X.
5. **ξ*-closed** [1] if α-cl(A) $\subseteq$ int(U) whenever A $\subseteq$ U, and U is a gα-open in X.
6. **ξ**-closed** [1] if α-cl(A) $\subseteq$ int cl (U) whenever A $\subseteq$ U, and U is a gα-open in X.
7. R**egular α-open** [6] if there is a regular open set U such that U $\subseteq$ A $\subseteq$ α-cl(U).
8. **rgα-closed** [6] if α-cl(A) $\subset$ U whenever A $\subset$ U and U is regularly α-open in X.

The complement of regular closed (resp. α-closed, gα-closed, ξ-closed, ξ*-closed, ξ* *-closed, rgα-closed, regular α-open) set is said to be **regular open** (resp. **α-open**, **gα-open**, **ξ-open, ξ*-open, ξ**-open, rgα-open, regularly α-closed**) set. The intersection of all ξ-closed subset of X containing A is called the **ξ-closure of A** and is denoted by **ξ-cl(A)**. The union of all ξ-open sets contained in A is called **ξ-interior of A** and is denoted by **ξ- int(A)**. The family of ξ-open (resp. ξ-closed) sets of a space X is denoted by **ξO(X)** (resp. **ξC(X)**).

2.2. Remark 1: Every α-closed (resp. α-open) set is ξ-closed (resp. ξ-open) set.

Definitions stated above, we have the following diagram:



However the converses of the above are not true may be seen by the following examples.

2.3. Example 1: Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X}. Then A = {c} is α-closed set as well as ξ, ξ* *-closed set but not closed set in X.

2.4. Example 2: Let X = {a, b, c, d, e} and τ = {ϕ, {a}, {d}, {e}, {a, d}, {a, e}, {d, e}, {a, d, e}, X}. Then the set A = {a, d, e} is rgα-closed set but not gα-closed set in X.

2.5. Example 3: Let X = {a, b, c, d} and τ = {ϕ, {a}, {b}, {a, b}, {a, b, c}, X}. Then the set A = {c} is gα-closed set but not closed set in X.

3. ξ-REGULAR SPACES

3.1. Definition 2: A topological space X is said to be **regular space** (resp. **α-regular** [2], **ξ-regular**) if for every closed set F and a point $x \notin F$, there exist disjoint open (resp. α-open, ξ-open) sets U, V of X such that $F \subset U$ and $x \in V$. Clearly every regular space is ξ-regular space.

3.2. Theorem 1: For a topological space X, the following are equivalent:

(a) X is regular.

(b) For each closed set F and each $x \in X - F$, there exists disjoint ξ-open sets U and V of X such that $x \in U$ and $F \subset V$.

Proof: It is obvious that (a) ⇒ (b), since every open set is ξ-open. (b) ⇒ (a), for any closed subset F of X and any point x in X – F, there exists disjoint ξ-open sets U and V of X such that $x \in U$ and $F \subset V$. Now, let G = int(cl(int(U))) and H = int(cl(int(V))). Then we have $x \in U \subset G$ and $F \subset V \subset H$. Since U and V are disjoint, G and H are disjoint open sets of X. This shows that X is regular.

3.3. Corollary 1: A topological space is ξ-regular if and only if it is regular.

3.4. Theorem 2: For a topological space X, the following are equivalent:

(a) X is ξ- regular.

(b) For every point x of X and every open set V containing x, there exists a ξ-open set U such that $x \in U \subset \xi\text{-cl}(U) \subset V$.

Proof: (a) ⇒ (b). Let $x \in X$ and V be an open set containing x. Then X – V is closed and $x \notin X - V$. By (a), there exist ξ -open sets U and S such that $x \in U$, $X - V \subset S$ and $U \cap S = \phi$. Since $x \in U \subset X - S \subset V$ and ξ-cl (X – S) = X – S. We obtain the required inclusions: $x \in U \subset \xi\text{-cl}(X - U) \subset V$.

(b) ⇒ (a). Let F be a closed set and a point $x \notin F$. Then, by (b) there exists a ξ -open set U such that $x \in U \subset \xi\text{-cl}(U) \subset X - F$. Therefore, U and X – ξ-cl(U) are the required ξ-open sets.

3.5. Definition 3: A function f: X →Y is said to be

1. **almost ξ-open** (resp. **ξ-open**) if f(U) is ξ -open in Y for every regular open (resp. open) set U of X.
2. **almost ξg-closed** (resp. **ξg-closed**) if f(F) is ξg-closed in Y for every regular closed (resp. closed) set F of X .
3. **pre ξ-closed** if f(F) is ξ -closed in Y for every ξ-closed set F of X.

3.6. Remark 2: For a function, we have the following implications.

closed ⇒ α - closed ⇒ ξ -closed ⇒ gξ -closed ⇒ almost ξg -closed

3.7. Lemma 1: A surjection f: X → Y is almost ξg-closed almost iff for each subset of Y and each $U \in RO(X)$ containing $f^{-1}(S)$, there exists a ξg-open set V of Y such that $S \subset V$ and $f^{-1}(V) \subset U$.

3.8. Theorem 3: If f: X → Y is an almost ξ - open almost ξg-closed continuous surjection and X is regular space, then Y is regular.

Proof: Let y be any point of Y and F be any closed of Y not containing y. There exists a point $x \in X$ with f(x) = y. Since X is regular and f is continuous, there exist disjoint open sets U and V of X such that $x \in U$ and $f^{-1}(F) \subset V$. Now, put G = int(cl(U)) and H = int(cl(V)), then we have $y \in f(U) \subset f(int(cl(U))) = f(G)$ and f(G) is ξ -open because int(cl(U)) ∈ RO(X) and f is almost ξ-open. Since $f^{-1}(F) \subset V \subset H$ and f is almost ξg-closed, by **Lemma 1**, there exists a ξg-open set W of Y such that $F \subset W$ and $f^{-1}(W) \subset H$. Since F is closed, we have F ⊂ ξ-int(W). Since G and H are disjoint, f(G) and ξ-int(W) are disjoint ξ -open sets of Y. It follows from **Theorem 1** that Y is regular.

3.9. Corollary 2: If f: X → Y is a continuous, ξ-open and ξg-closed surjection and X is a regular space, then Y is ξ-regular.

3.10. Corollary 3: If f: X → Y is a continuous, pre ξ-closed bijection and X is ξ-regular space, then Y is ξ-regular.

Proof: Every pre ξ -closed function is ξ-closed and hence almost ξg-closed. Since f is bijective, it is almost ξ - open and the proof follows from **Theorem 3** and **Corollary 1**.

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