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Cost Benefit Analysis of Three Unit Redundant System Model with Correlated Failures and Repairs

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In this paper, an attempt is made to study an analysis of three unit redundant system model with correlated failures and repair times. In this work analysis of reliability and mean time to system failure, availability analysis, busy period analysis and profit function analysis are studied. Also graphical study of the system model is shown. The result indicated that higher correlation between the failure and repair times provides the better system performance.

Keywords: Mean Time to System Failure (MTSF), Steady State Availability, Busy Period, Modified Bessel's Function, Bivariate Exponential, Laplace Transform and L-Hospital's Rule.

1. INTRODUCTION

Many authors like Nakagava[1], Bhardwaj and Chander[2,3], Malik and Bhardwaj[4] discussed reliability to analyse the complex and priority redundant system models under different sets of assumptions, using supplementary variable and regenerative point techniques. In all these models it is assumed that the failure and repair times are uncorrelated random variables. But in practice it seems to be unrealistic because in many cases failure and repair times may be found correlated. Keeping this fact in view, Goel, Sharma and Gupta[5] introduced the concept of correlation in a single server two-unit cold standby system subject to some stringent assumptions. Further, Murari and Goyal[6], Singh[7] used the concept of correlation in the analysis of two-unit system models, but not much work is done by using this concept in three unit and complex system models, we investigate a three unit complex system with correlated failure and repair time distributions.

This model analyses a single server three unit complex system assuming the repair discipline to be first come first serve (FCFS) and the repair of a failed unit is completed without any interruption. The later failed unit waits for repair till the repair of the unit already in hand is completed. Three units are non-identical and named as unit A, B and C. Initially all the units are good and operative. Units are arranged in such a way that the system failure occurs if either unit A or both the units B and C fail totally. The configuration of the system model with units A, B and C is shown in Fig. 1.

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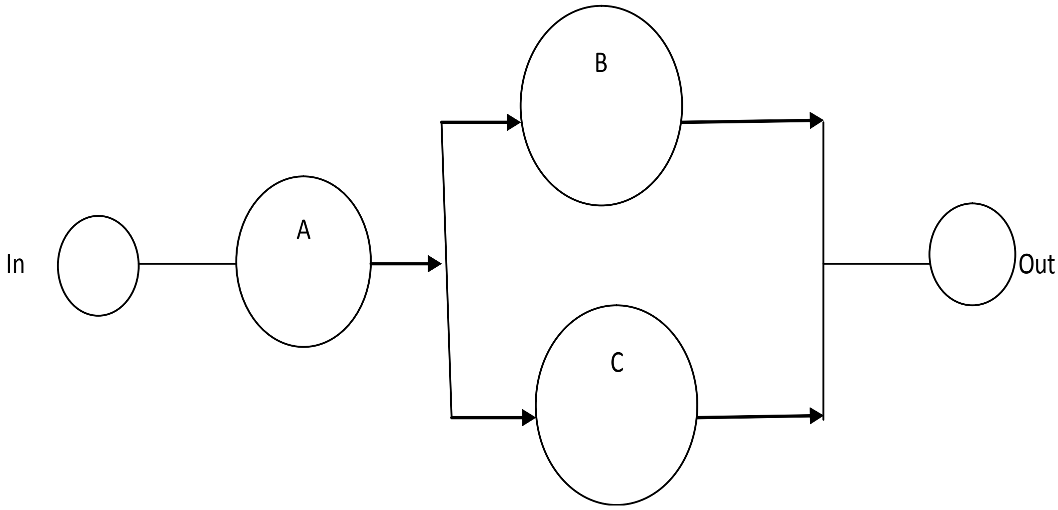

Fig. 1: Configuration of the System Model

The joint distribution of failure and repair times of each unit is taken to be bivariate exponential having the density function of the following form:

$$f_i(x, y) = \alpha_i \beta_i (1 - r_i)\ e^{-(\alpha_i x + \beta_i y)} I_0\left(2\surd\ (\alpha_i \beta_i r_i x\, y)\right)$$

$$\alpha_i,\ \beta_i,\ x,\ y > 0,\ 0 \le r_i \le 1,\ i = 1, 2$$

where, $I_0\left(2\surd\ (\alpha_i \beta_i r_i x\, y)\right)$ is the modified Bessel's function of type one and order zero and is given by:

$$I_0\left(2\surd\ (\alpha\ \beta\ r\, x\, y)\right) = \sum_{K=0}^{\infty} \left\{(\alpha\beta r x y)^K\right\} / (K!)^2$$

Using regenerative point technique the following important measures of system effectiveness have been obtained:

(i) Reliability and mean time to system failure (MTSF).
(ii) Point wise and steady state availabilities of the system.
(iii) Expected up time of the system and expected busy period of the repairman during a finite interval of time.
(iv) Net expected profit incurred by the system during (0, t) and in steady state.

2. SYSTEM DESCRIPTION AND ASSUMPTIONS

The system is analyzed under following assumptions:

(i) System consists of three non-identical units A, B and C. For successful operation of the system unit A and at least one of the units B and C should function.
(ii) System fails when either unit A or both the units B and C fail.
(iii) A single repair facility is available to repair failed units.
(iv) Service discipline is FCFS.
(v) A repaired unit's works as good as new.

3. NOTATION AND STATES OF THE SYSTEM

For the system model we have following notations:

X_i, Y_i (i=1, 2, 3)	:	random variables denoting the failure and repair times for units A, B and C respectively.
$f_i(x, y)$	:	joint p.d.f. of X_i, Y_i.
$g_i(x)$	:	marginal p.d.f. of X_i
		$= \alpha_i(1 - r_i)e^{-\alpha_i(1-r_i)x}$
©	:	complex varibility.
$k_i(y\|x)$	:	conditional probability density function of Y_i given X_i=x
		$= \beta_i e^{-(\beta_i y + \alpha_i r_i x)} I_0\left(2\sqrt{\ (\alpha_i\beta_i r_i x\, y)}\right)$.
$K_i(y\|x)$	:	conditional c.d.f. of Y_i given X_i=x .
A_g, A_o, A_r, A_R, A_w	:	unit A is good, operative, under repair, repair continued from earlier state and waiting for repair respectively.

Similarly we may describe the symbols for units B and C by taking B_g, B_o, B_r, B_R, B_w and C_g, C_o, C_r, C_R, C_w. With the help of the above symbols possible states of the system model are:

UP States	DOWN/FAILED States
$S_0 = (A_o, B_o, C_o)$	$S_3 = (A_r, B_g, C_g)$
$S_1 = (A_o, B_r, C_o)$	$\underline{S_4} = (A_g, B_R, C_w)$
$S_2 = (A_o, B_o, C_r)$	$\underline{S_5} = (A_g, B_w, C_R)$
	$\underline{S_6} = (A_w, B_g, C_R)$
	$S_7 = (A_w, B_R, C_g)$

The underlined states are non regenerative. Transition diagram along with all transitions is shown in Fig. 2.

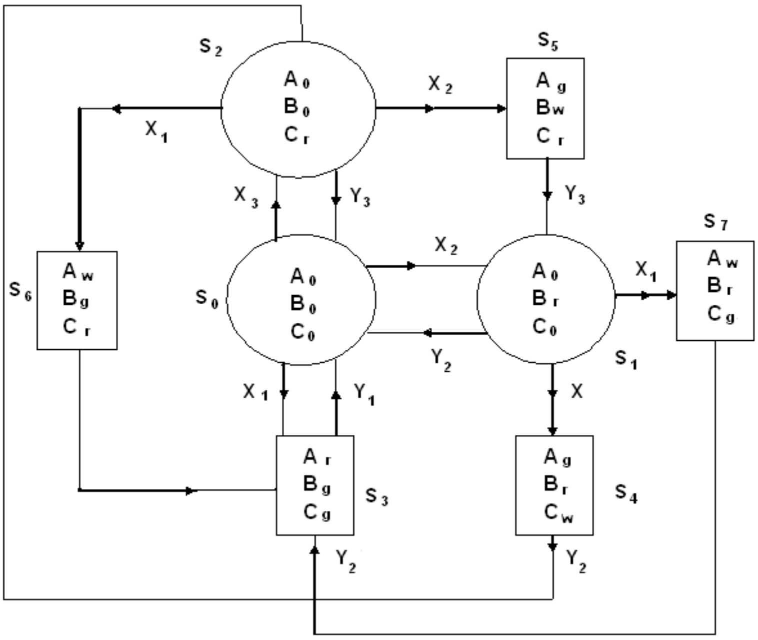

Fig. 2: Transition Diagram

4. ANALYSIS OF RELIABILITY AND MEAN TIME TO SYSTEM FAILURE

Let the random variable T_i denotes the ime to system failure when the system starts from the state $S_i \in E$ (i = 0, 1, 2, 4) then the reliability of the system is given by:

$$R_i(t) = P(T_i > t)$$

To determine $R_i(t)$, we regard the failed state S_3, S_4, S_5, S_6, S_7 of the system as absorbing states. By simple probabilistic reasoning, we have the following relations:

$$\begin{aligned} R_0(t) &= Z_0(t) + q_{01}(t) © R_1(t) + q_{02}(t) © R_2(t) \\ R_1(t) &= Z_1(t) + q_{10}(t) © R_0(t) \\ R_2(t) &= Z_2(t) + q_{20}(t) © R_0(t) \end{aligned} \qquad \text{-(1-3)}$$

$$Z_0 = \exp[-\sum_{i=1}^{3} \alpha_i (1 - r_i) t]$$

where, $Z_1(t) = e^{-[\alpha_1(1-r_1) + \alpha_3(1-r_3)]t} \overline{K_2}(t \mid x)$

$$Z_2(t) = e^{-[\alpha_1(1-r_1) + \alpha_2(1-r_2)]t} \overline{K_3}(t \mid x)$$

Taking Laplace transforme of the relations (1-3) and solving the resulting set of equations for $R_0^*(s)$, we get:

$$R_0^*(s) = N_1(s)/D_1(s) \qquad -(4)$$

where, $N_1(s) = Z^*_0 + q^*_{01}Z^*_1 + q^*_{02} Z^*_2$, and $D_1(s) = 1- q^*_{01} q^*_{10} - q^*_{02} q^*_{20}$.

Taking inverse Laplace trancforme of equation (4), we get the reliability of the system. To get MTSF, we have the following equation:

$$E(T_0) = \int_0^\infty R_0(t)\, dt = \lim_{s\to 0} R^*_0(s) = N_1(0)/D_1(0) \qquad -(5)$$

where, $N_1(0) = \psi_0 + p_{01}\psi_1 + p_{02}\psi_2$ and $D_1(0) = 1- p_{01}p_{10} - p_{02}p_{20}$. Here, we have used the results $q^*_{ij}(0) = p_{ij}$ and $Z^*_i(0) = \psi_i$.

5. AVAILABILITY ANALYSIS

Let us define $A_i(t)$ as the probability that the system is up at epoch t when it initially started operation from regenerative state S_i. Using the definition of $A_i(t)$ and probabilistic concepts, the following relations are obtained among $A_i(t)$, (where i = 0, 1, 2, 3)

$$A_0(t) = Z_0(t) + q_{01}(t) © A_1(t) + q_{02}(t) © A_2(t) + q_{03}(t) © A_2(t)$$
$$A_1(t) = Z_1(t) + q_{10}(t) © A_0(t) + q^{(4)}_{12}(t) © A_2(t) + q^{(7)}_{13}(t) © A_3(t)$$
$$A_2(t) = Z_2(t) + q_{20}(t) © A_0(t) + q^{(5)}_{21}(t) © A_1(t) + q^{(6)}_{23}(t) © A_3(t)$$
$$A_3(t) = q_{30}(t) © A_0(t) \qquad -(6\text{-}9)$$

Taking Laplace transforme of the above set of equations and solving for $A^*_o(s)$, we get:

$$A^*_o(s) = N_2(s) / D_2(s) \qquad -(10)$$

where, $N_2(s) = Z^*_0(1-q^{(4)*}_{12} q^{(5)*}_{21}) + Z^*_1(q^*_{01} + q^*_{02}q^{(5)*}_{21}) + Z^*_2(q^*_{02} + q^*_{01} q^{(4)*}_{12})$ and

$$D_2(s) = (1-q^*_{03}q^*_{30})(1-q^{(4)*}_{12} q^{(5)*}_{21}) - q^*_{01}[q^*_{10} + q^{(4)*}_{12} q^*_{20} + q^*_{30}(q^{(7)*}_{13} + q^{(4)*}_{12} q^{(6)*}_{23})] - q^*_{02}[q^*_{20} + q^*_{10} q^{(5)*}_{21} + q^*_{30}(q^{(6)*}_{23} + q^{(5)*}_{21} q^{(7)*}_{13})] \qquad -(11)$$

The steady state probability that the system will be up in the long run is given by

$$A_0 = \lim_{t\to 0} A_0(t) = \lim_{s\to 0} A^*_0(s) = N_2(0)/ D_2(0) \qquad -(12)$$

Now using the result

$\lim_{s\to 0} Z^*_i(s) = \Psi_i$ and $\lim_{s\to 0} q^*_{ij}(s) = p_{ij}$, we get

$$N_2(0) = \Psi_0(1- p^{(4)}_{12} p^{(5)}_{21}) + \psi_1(p_{01} + p_{02} p^{(5)}_{21}) + \Psi_2(p_{02} + p_{01} p^{(4)}_{12}) \qquad -(13)$$

and,

$$\begin{aligned} D_2(0) &= (1-p_{03})(1-p^{(4)}_{12} p^{(5)}_{21}) - p_{01}[p_{10} + p^{(4)}_{12} p_{20} + p^{(7)}_{13} + p^{(4)}_{12}p^{(6)}_{23}] \\ &\quad - p_{02}[p_{20} + p_{10}p^{(5)}_{21} + p^{(6)}_{23} + p^{(5)}_{21} p^{(7)}_{13}] \\ &= (1-p_{03})(1-p^{(4)}_{12} p^{(5)}_{21}) - p_{01}[(p_{10} + p^{(7)}_{13}) + p^{(4)}_{12}(p_{20} + p^{(6)}_{23})] \\ &\quad - p_{02}[(p_{20} + p^{(6)}_{23}) + p^{(5)}_{21}(p_{10} + p^{(7)}_{13})] \\ &= (1-p_{03})(1-p^{(4)}_{12} p^{(5)}_{21}) - p_{01}[1- p^{(4)}_{12} + p^{(4)}_{12}(p_{20} + p^{(6)}_{23})] \\ &\quad - p_{02}[1- p^{(5)}_{21} + p^{(5)}_{21}(p_{10} + p^{(7)}_{13})] \\ &= (1-p_{03})(1-p^{(4)}_{12} p^{(5)}_{21}) - p_{01}[1- p^{(4)}_{12} p^{(5)}_{21}] - p_{02}[1- p^{(4)}_{12} p^{(5)}_{21}] \end{aligned}$$

$= (1-p_{03})(1-p^{(4)}_{12}\, p^{(5)}_{21}) - (p_{01} + p_{02})[1- p^{(4)}_{12}\, p^{(5)}_{21}]$
$= (1-p_{03})(1-p^{(4)}_{12}\, p^{(5)}_{21}) - (1- p_{03})[1- p^{(4)}_{12}\, p^{(5)}_{21}]$
$= 0$

Therefore, by using L' Hospital's rule

$$A_0 = N_2 (0) / D'_2 (0) \qquad \text{-(14)}$$

To obtain D'_2 (0), we use the relation $q^*_{ij}(0) = - m_{ij}$ and $q^{(k)*}_{ij}(0) = - m^{(k)}_{ij}$ and collect the coefficients of m_{ij}'s in D'_2 (0) as follows:

Coefficient of m_{01} $= p_{10} + p^{(7)}_{13} + p^{(4)}_{12}(p_{20} + p^{(6)}_{23}) = 1- p^{(4)}_{12}\, p^{(5)}_{21}$
Coefficient of m_{20} $= p_{20} + p^{(6)}_{23} + p^{(5)}_{21}(p_{10} + p^{(7)}_{13}) = 1- p^{(4)}_{12}\, p^{(5)}_{21}$
Coefficient of m_{03} $= 1- p^{(4)}_{12}\, p^{(5)}_{21}$
Coefficient of m_{10} $= p_{01} + p_{02}\, p^{(5)}_{21}$
Coefficient of $m^{(4)}_{12}$ $= (1- p_{03})p^{(5)}_{21} + p_{01}(p_{20} + p^{(6)}_{23})$
$= (1- p_{03})p^{(5)}_{21} + p_{01}(1- p^{(5)}_{21})$
$= p_{01} + p_{02}\, p^{(5)}_{21}$
Coefficient of $m^{(7)}_{13}$ $= p_{01} + p_{02}\, p^{(5)}_{21}$
Coefficient of m_{20} $= p_{02} + p_{01}\, p^{(4)}_{12}$
Coefficient of $m^{(5)}_{21}$ $= (1- p_{03})p^{(4)}_{12} + p_{02}(p_{10} + p^{(7)}_{13})$
$= p_{02} + p_{01}\, p^{(4)}_{12}$
Coefficient of $m^{(6)}_{23}$ $= p_{02} + p_{01}\, p^{(4)}_{12}$
Coefficient of m_{30} $= 1-p^{(4)}_{12}\, p^{(5)}_{21}$

Therefore,

$$D'_2 (0)= (m_{01} + m_{02} + m_{03}) (1-p^{(4)}_{12}\, p^{(5)}_{21}) + (m_{10} + m^{(4)}_{12} + m^{(7)}_{13}) (p_{01} + p_{02}\, p^{(5)}_{21}) + (m_{20} + m^{(5)}_{21} + m^{(6)}_{23}) (p_{02} + p_{01}p^{(4)}_{12}) + m_{30} (1-p^{(4)}_{12}\, p^{(5)}_{21})$$
$$= (\mu_0 + \mu_3) (1-p^{(4)}_{12}\, p^{(5)}_{21} + n_1 (p_{01} + p_{02}\, p^{(5)}_{21}) + n_2 (p_{02} + p_{01}\, p^{(4)}_{12}) - \mu_3 [p_{01} (p_{10} + p^{(4)}_{12}\, p_{20}) + p_{02} (p_{20} + p_{01}\, p^{(5)}_{21})] \qquad \text{-(15)}$$

where, $n_1 = \mu_1 + (1-p_{10})\, \mu_4$ and $n_2 = \mu_2 + (1-p_{20})\, \mu_5$

The expected up time of the system during (0, t) is given by:

$$\mu_{up}(t) = \int_0^t A_0(u)\, du$$

so that, $$\mu^*_{up}(s) = A^*_0(s)/s \qquad \text{-(16)}$$

6. BUSY PERIOD ANALYSIS

Define B_i (t) as the probability that the system having started from regenerative state $S_i \in E$, at time t = 0, is under repair at instant 't'. The expression for B_i (t) (i = 0, 1, 2, 4, 5, 6) can be written as under:

$B_0 (t) = q_{01}(t) © B_1(t) + q_{02}(t) © B_2(t) + q_{03}(t) © B_3(t)$

$B_1(t) = Z_1(t) + q_{10}(t) © B_0(t) + q^{(7)}_{13}(t) © B_3(t) + q_{17}(t) © Z_7(t) + q^{(4)}_{12}(t) © B_2(t) + q_{14}(t) © Z_4(t)$

$B_2(t) = Z_2(t) + q_{20}(t) © B_0(t) + q^{(5)}_{21}(t) © B_1(t) + q_{25}(t) © Z_5(t) + q^{(6)}_{23}(t) © B_3(t) + q_{26}(t) © Z_6(t)$

$B_3(t) = Z_3(t) + q_{30}(t) © B_0(t)$ -(17-20)

Taking Laplace transforme of the relations (17-20), we get:

$B^*_0(s) = q^*_{01}(s) B^*_1(s) + q^*_{02}(s) B^*_2(s) + q^*_{03}(s) B^*_3(s)$

$B^*_1(s) = Z^*_1(s) + q^*_{10}(s) B^*_0(s) + q^{(7)*}_{13}(s) B^*_3(s) + q^*_{17}(s) Z^*_7(s) + q^{(4)*}_{12}(s) B^*_2(s) + q^*_{14}(s) Z^*_4(s)$

$B^*_2(s) = Z^*_2(s) + q^*_{20}(s) B^*_0(s) + q^{(5)*}_{21}(s) B^*_1(s) + q^*_{25}(s) Z^*_5(s) + q^{(6)*}_{23}(s) B^*_3(s) + q^*_{26}(s) Z^*_6(s)$

$B^*_3(s) = Z^*_3(s) + q^*_{30}(s) B^*_0(s)$ -(21-24)

Solving the above equations for $B^*_0(s)$, we get:

$$B^*_0(s) = N_3(s) / D_2(s) \quad -(25)$$

where, $N_3(s) = (q^*_{01} + q^*_{02} q^{(5)*}_{21})(Z^*_1 + q^*_{14} Z^*_4 + q^*_{17} Z^*_7) + (q^*_{02} + q^*_{01} q^{(4)*}_{12})(Z^*_2 + q^*_{23} Z^*_3 + q^*_{26} Z^*_6)$
$+ Z^*_3 [q^*_{01}(q^{(7)*}_{13} + q^{(4)*}_{12} q^{(6)*}_{23}) + q^*_{02}(q^{(6)*}_{23} + q^{(7)*}_{13} q^{(5)*}_{21}) + q^*_{03}(1 - q^{(4)*}_{12} q^{(5)*}_{21})]$

and $D_2(s)$ is given by equation (11).

In the long run the expected fraction of the time for which the repairman is busy in the repair of the system is given by:

$$B_0 = \lim_{t\to\infty} B_0(t) = \lim_{s\to\infty} s\, B^*_0(s) = N_3(0) / D'_2(0) \quad -(26)$$

where, $N_3(0) = (p_{01} + p_{02} p^{(5)}_{21})(\Psi_1 + p_{14} \Psi_4 + p_{17} \Psi_7) + (p_{02} + p_{01} p^{(4)}_{12})(\Psi_2 + p_{25} \Psi_5 + p_{26} \Psi_6)$
$+ \Psi_3 [p_{01}(p^{(7)}_{13} + p^{(4)}_{12} p^{(6)}_{23}) + p_{02}(p^{(6)}_{23} + p^{(7)}_{13} p^{(5)}_{21}) + p_{03}(1 - p^{(4)}_{12} p^{(5)}_{21})]$ -(27)

Thus using (27) and (26), we get expression for B_0. The expected busy period of the repairman during (0, t) is given by:

$$\mu_b(t) = \int_0^t B_0(u)\,du$$

so that, $$\mu^*_b(s) = B^*_0(s) / s \quad -(28)$$

7. PROFIT FUNCTION ANALYSIS

Two profit functions $P_1(t)$ and $P_2(t)$ can be found easily for the system model under study with the help of the characteristics obtained earlier. The expected total profits incurred during (0, t) are:

$P_1(t)$ = expected total revenue in (0, t) – expected total expenditure in (0,t)

$$= K_0 \mu_{up}(t) - K_1 \mu_b(t) \quad -(29)$$

and $$P_2(t) = K_0 \mu_{up}(t) - K_2 V_0(t) \quad -(30)$$

where K_0 is the revenue per unit up time and K_1 and K_2 are the amounts paid to the repairman per unit of time when he is busy in repairing the failed unit and per unit repair cost respectively.

The expected profits per unit time in steady state are given by:

$$P_1 = \lim_{t \to 0} [P_1(t)/t] = K_0A_0 - K_1B_0 \qquad -(31)$$

and

$$P_2 = \lim_{t \to 0} [P_2(t)/t] = K_0A_0 - K_1V_0 \qquad -(32)$$

8. GRAPHICAL STUDY OF THE SYSTEM MODEL

For a more concrete study of the MTSF and profit functions, we have plotted these characteristics in Fig. 3 and Fig. 4 respectively w.r.t. α_1 for three different values of r_1 as 0.25, 0.50, 0.75 while the other parameters are fixed as α_2 = 0.05, α_3 = 0.10, $r_2 = r_3$ = 0.5, β_1= 0.20, β_2 = 0.40, K_0 = 300, K_1= 100, K_2 = 50.

From Fig. 3, it is clear that the MTSF slowly decreases almost linearly with the increase in α_1 and it increases as the coefficient of correlation r_1 increases. Also from Fig. 4(a) and 4(b), it is evident that the cost function decreases exponentially as α_1 increases and it increases with the increase in r_1.

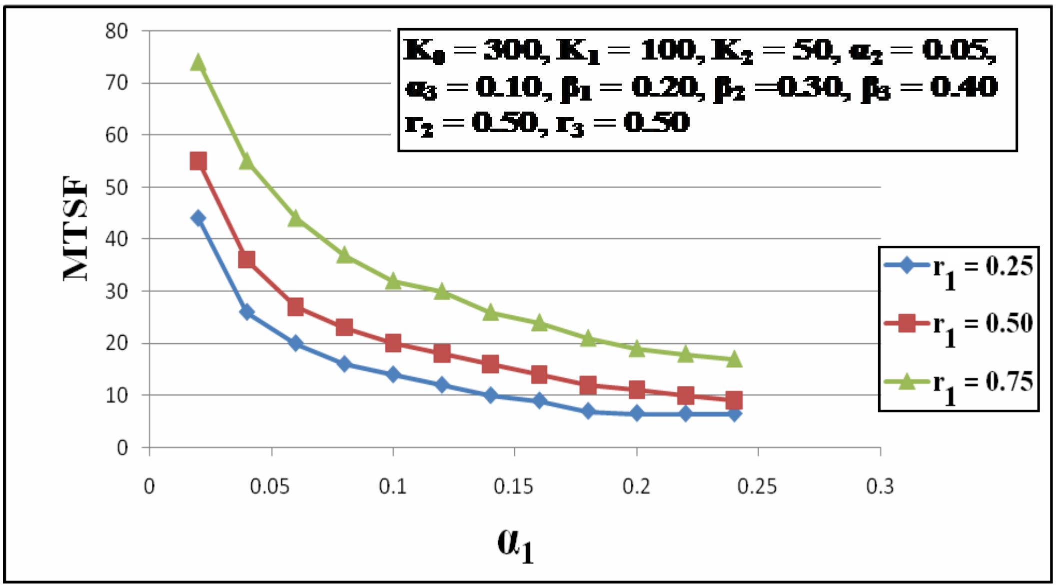

Fig. 3: MTSF with α_1

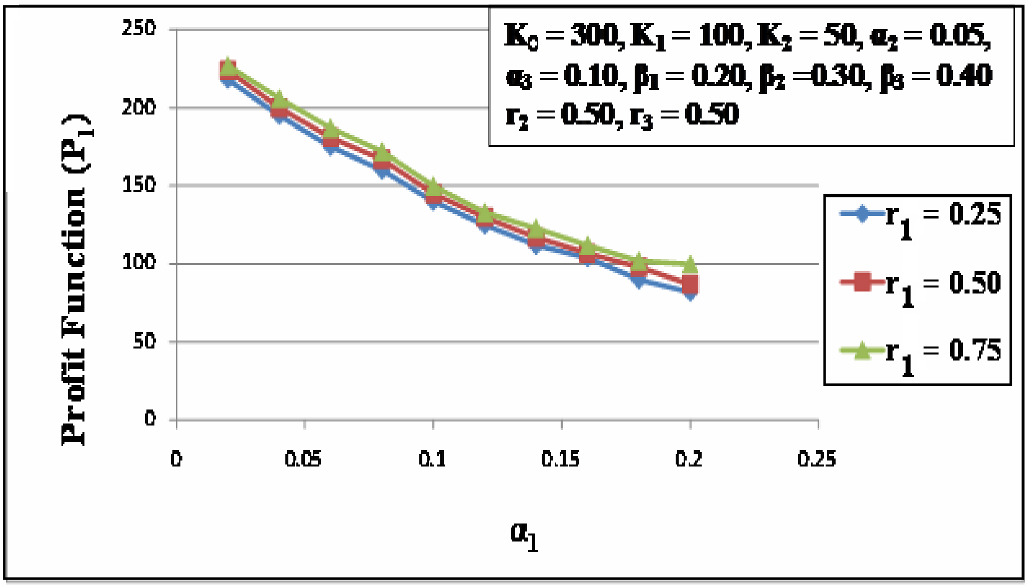

Fig. 4(a): Profit Function $P_1(t)$ with α_1

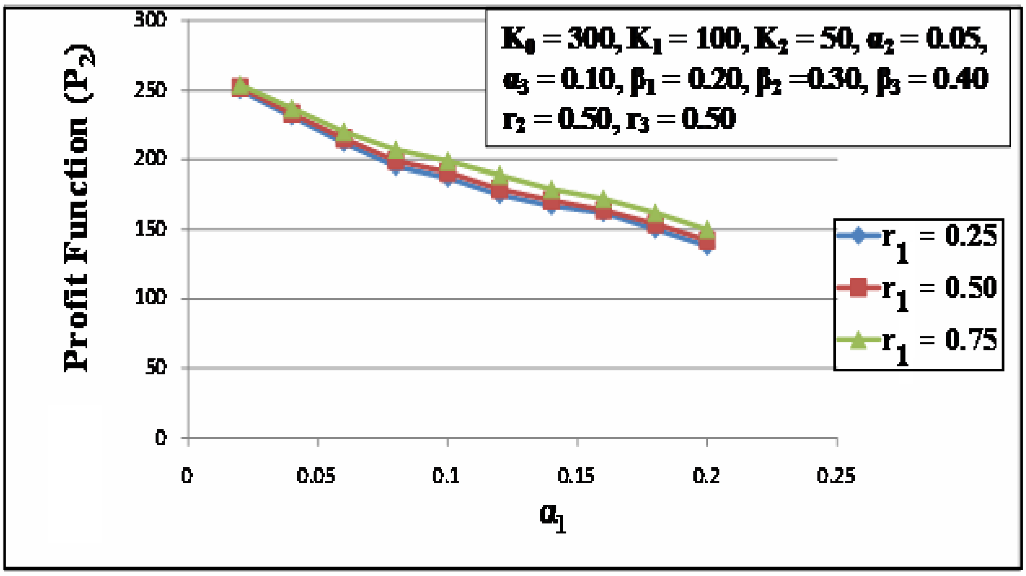

Fig. 4(b): Profit Function $P_2(t)$ with α_1

9. CONCLUSION

We concluded that higher correlation between the failure and repair times provides the better system performance.

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