Some Exact Solutions for Rotating Flows of a Generalized Second Grade Fluid in Cylindrical Domains

Rakhi Sharma[1], A. K. Bhargava[2], Narottam Kumar[3] and A. B. Chandramouli[4]
[1]Research Scholar, C.C.S. University, Meerut
[2]Associate Professor, Department of Mathematics, M.M.H. College, Ghaziabad (UP), India
[3,4]Associate Professor, Department of Mathematics, Meerut College, Meerut (UP), India
[1]E-mail:rakhi_shrma83@yahoo.com,[2]E-mail: dr.bhargavaak@gmail.com

In this paper, we have studied the rotational flow of a generalized second grade fluid between two infinite coaxial circular cylinders. The velocity field and the shear stress obtained by means of Laplace and Hankel transforms are presented under series form in terms of generalized functions. At time $t = 0$, the fluid and cylinders are at rest. At $t = 0^+$, both cylinders suddenly begin to rotate, about their common axis, with a constant angular acceleration. The obtained solutions can be specialized to give the similar solutions for ordinary second grade and Newtonian fluids performing the same motion.

Keywords: Generalized Second Grade Fluid, Laplace and Hankel Transforms, Rotational Flow, Shear Stress, Velocity Field.

1. INTRODUCTION

Many materials such as drilling mud, certain oils and greases, blood and many emulsions have been used as non-Newtonian fluids. Amongst the many models which have been treated as non-Newtonian behavior, the fluids of differential type have received special attention. The second grade fluids are the common non-Newtonian viscoelastic fluids in industrial fields, such as polymer solutions.

Several authors [17,18] suggested that the integer-order models for viscoelastic material seen to be inadequate from both qualitative and quantitative point of view. At the same time, they proposed fractional-order laws of deformation for modeling the viscoelastic behavior of real materials. Caputo and Mainardi [3] formulated the fractional-order model which includes the classical law. Bagley and Torvik [1,2] and Rogers [15] have shown that classical law is very useful for modeling of most viscoelastic materials. In addition to experimental findings, they proved that the four-parameter model seems to be satisfactory for most real materials. Bagley and Torvik [1,2], Koeller [12], Xu and Tan [22,23] proposed the fractional derivative approach to viscoelasticity in order to describe the properties of numerous viscoelastic materials. The Rouse [16] theory provides us the presence of fractional derivative along with the first derivative of classical viscoelasticity in the relation between stress and strain for some polymers. Ferry et al. [6] modified the Rouse theory in concentrated polymer solutions and polymer solids with no cross-linking.

Thus, the fractional calculus approach to viscoelasticity for the study of viscoelastic material properties is justified, at least for polymer solutions and for polymer solids without cross-linking.

2. FLOW BETWEEN COAXIAL CYLINDERS

Among many constitutive assumptions that have been employed to study non-Newtonian fluid behavior, are class that has gained support from both the experimentalists and theoreticians is that of Rivlin-Ericksen fluids of second grade. The Cauchy stress tensor **T** for such fluids is given by [7,14] as follows

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $-p$ is the pressure, $\mathbf{I}$ is the unit tensor, μ is the coefficient of viscosity, α_1 and α_2 are the normal stress moduli and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the kinematic tensors defined by

$$\mathbf{A}_1 = grad\,\mathbf{v} + \left(grad\,\mathbf{v}\right)^T, \tag{2}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1\left(grad\,\mathbf{v}\right) + \left(grad\,\mathbf{v}\right)^T\mathbf{A}_1, \tag{3}$$

where d/dt denote the material time derivative, $\mathbf{v}$ is the velocity field, the superscript T denote the transpose operation and “grad” is the gradient operator.

If the second grade fluid, given by Eq. (1) is compatible with thermodynamics, then the material moduli must meet the following restrictions [5]

$$\mu \geq 0,\ \alpha_1 \geq 0\ and\ \alpha_1 + \alpha_2 = 0. \tag{4}$$

For a generalized second grade fluid, Eq. (1) still holds but $\mathbf{A}_2$ is defined as follows [1,8,11,19]

$$\mathbf{A}_2 = D_t^\beta\mathbf{A}_1 + \mathbf{A}_1\left(grad\,\mathbf{v}\right) + \left(grad\,\mathbf{v}\right)^T\mathbf{A}_1, \tag{5}$$

and D_t^β is the Riemann-Liouville fractional calculus operator [4]. For $\beta = 1$, we have $D_t^1 f(t) = df(t)/dt$ and hence Eq. (5) is reduced to Eq. (3).

2.1. Governing Equation

Let us consider an incompressible second grade fluid at rest in an annular region between two straight circular cylinders of radii R_1 and R_2 ($R_2 > R_1$). At time t = 0, the fluid and cylinders are at rest. At time t = 0^+, both cylinders suddenly begin to rotate, about their common axis (r = 0), with constant angular accelerations. Owing to the shear, the fluid between the cylinders is gradually moved and its velocity in cylindrical coordinated (r,θ,z) is given by [7,8,20]

$$\mathbf{v}=\mathbf{v}(r,t)=v(r,t)\mathbf{e}_\theta, \tag{6}$$

where $\mathbf{e}_\theta$ is the transverse unit vector along θ-direction. For such flows, the constraint of incompressibility is automatically satisfied.

Based on the above suppositions, constitutive equation of generalized second grade fluid, corresponding to this motion is

$$\tau(r,t)=\left(\mu+\alpha_1 D_t^\beta\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)v(r,t), \tag{7}$$

where $\tau(r,t)=S_{r\theta}(r,t)$ is the shear stress, which is non-zero.

In absence of body forces and a pressure gradient in the axial direction, the balance of linear momentum leads to relevant equation:

$$\rho\frac{\partial v(r,t)}{\partial t}=\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\tau(r,t), \tag{8}$$

where ρ is the constant of density of the fluid.

Eliminating $\tau(r,t)$ between Eq. (7) and (8), we get the governing differential equation of our problem as follows

$$\frac{\partial v(r,t)}{\partial t}=\left(\upsilon+\alpha D_T^\beta\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)v(r,t),\ r\in(R_1,R_2),\ t>0, \tag{9}$$

where $\upsilon=\mu/\rho$ is the kinematic viscosity of the fluid and $\alpha=\alpha_1/\rho$ is the fractional viscoelastic constant. Consequently, the velocity field corresponding to this motion does not depend upon the material module α_2.

The appropriate initial and boundary conditions are

$$v(r,0)=0;\quad r\in(R_1,R_2), \tag{10}$$

$$v(R_1,t)=R_1\Omega_1 t,\quad v(R_2,t)=R_2\Omega_2 t\quad for\quad t>0. \tag{11}$$

In order to solve this problem, we shall use the Laplace and Hankel transforms as in [9,20,21].

2.2. Calculation of the Velocity Field

Applying the Laplace transforms to Eq. (9) and having the initial and boundary conditions (10) and (11) in mind, we obtain the following ordinary differential equation [10]

$$\left(\upsilon+\alpha q^{\beta}\right)\left[\frac{\partial^{2}\bar{v}(r,q)}{\partial r^{2}}+\frac{1}{r}\frac{\partial\bar{v}(r,q)}{\partial r}-\frac{\bar{v}(r,q)}{r^{2}}\right]-q\,\bar{v}(r,q)=0, \qquad (12)$$

where the image function $\bar{v}(r,q)=\int_0^{\infty}v(r,t)e^{-qt}dt$ of $v(r,t)$ has to satisfy the conditions

$$\bar{v}(R_1,q)=\frac{R_1\Omega_1}{q^2},\quad \bar{v}(R_2,q)=\frac{R_2\Omega_2}{q^2}, \qquad (13)$$

and q is the Laplace transform parameter.

Let us denote the finite Hankel transform of $\bar{v}(r,q)$ by

$$\bar{v}_n(q)=\int_{R_1}^{R_2}r\,\bar{v}(r,q)B_1(rr_n)\,dr,\, n=1,2,3,..., \qquad (14)$$

where $\quad B_1(rr_n)=J_1(rr_n)Y_1(R_2r_n)-J_1(R_2r_n)Y_1(rr_n),$

and r_n are the positive roots of the transcendental equation $B_1(R_1r)=0$ and $J_1(\cdot)$ and $Y_1(\cdot)$ are Bessel functions of order one of the first and second kind, respectively.

Applying the finite Hankel transform to Eq. (12), taking into account the conditions (13) and using the following relations

$$\frac{d}{dr}\left[B_1(rr_n)\right]=r_n\left[J_0(rr_n)Y_1(R_2r_n)-J_1(R_2r_n)Y_0(rr_n)\right]-\frac{1}{r}B_1(rr_n), \qquad (15)$$

$$J_0(z)Y_1(z)-J_1(z)Y_0(z)=-\frac{2}{\pi z}, \qquad (16)$$

we find that $\quad \left(\upsilon+\alpha q^{\beta}\right)\left\{\frac{2\left[R_2\Omega_2J_1(R_1r_n)-R_1\Omega_1J_1(R_2r_n)\right]}{\pi q^2J_1(R_1r_n)}-r_n^2\bar{v}_H(r_n,q)\right\}$

$$-q\bar{v}_H(r_n,q)=0,$$

or equivalently,

$$\overline{v}_H\left(r_n,q\right)=\frac{2\left[R_2\Omega_2J_1\left(R_1r_n\right)-R_1\Omega_1J_1\left(R_2r_n\right)\right]}{\pi J_1\left(R_1r_n\right)}\frac{\upsilon+\alpha q^{\beta}}{q^2\left[q+\alpha r_n^2q^{\beta}+\upsilon r_n^2\right]}. \quad (17)$$

Eq. (17) can be also written in the following equivalent form

$$\overline{v}_H\left(r_n,q\right)=\overline{v}_{1H}\left(r_n,q\right)+\overline{v}_{2H}\left(r_n,q\right), \quad (18)$$

where
$$\overline{v}_{1H}\left(r_n,q\right)=\frac{2}{\pi r_n^2}\frac{R_2\Omega_2J_1\left(R_1r_n\right)-R_1\Omega_1J_1\left(R_2r_n\right)}{J_1\left(R_1r_n\right)}\frac{1}{q^2}, \quad (19)$$

and
$$\overline{v}_{2H}\left(r_n,q\right)=-\frac{2}{\pi r_n^2}\frac{R_2\Omega_2J_1\left(R_1r_n\right)-R_1\Omega_1J_1\left(R_2r_n\right)}{J_1\left(R_1r_n\right)}\frac{1}{q\left[q+\alpha r_n^2q^{\beta}+\upsilon r_n^2\right]}. \quad (20)$$

The inverse Hankel transforms of the function $\overline{v}_{1H}$ and $\overline{v}_{2H}$ are

$$\overline{v}_1\left(r,q\right)=\frac{\Omega_1R_1^2\left(R_2^2-r^2\right)+\Omega_2R_2^2\left(r^2-R_1^2\right)}{\left(R_2^2-R_1^2\right)r}\frac{1}{q^2}, \quad (21)$$

and
$$\overline{v}_2\left(r,q\right)=-\pi\sum_{n=1}^{\infty}\frac{J_1\left(R_1r_n\right)\left[R_2\Omega_2J_1\left(R_1r_n\right)-R_1\Omega_1J_1\left(R_2r_n\right)\right]B_1\left(rr_n\right)}{J_1^2\left(R_1r_n\right)-J_1^2\left(R_2r_n\right)}$$
$$\times\frac{1}{q\left[q+\alpha r_n^2q^{\beta}+\upsilon r_n^2\right]}. \quad (22)$$

respectively.

Consequently, the function $\overline{v}\left(r,q\right)$ has the form

$$\overline{v}\left(r,q\right)=\frac{\Omega_1R_1^2\left(R_2^2-r^2\right)+\Omega_2R_2^2\left(r^2-R_1^2\right)}{\left(R_2^2-R_1^2\right)r}\frac{1}{q^2}$$
$$-\pi\sum_{n=1}^{\infty}\frac{J_1\left(R_1r_n\right)\left[R_2\Omega_2J_1\left(R_1r_n\right)-R_1\Omega_1J_1\left(R_2r_n\right)\right]B_1\left(rr_n\right)}{J_1^2\left(R_1r_n\right)-J_1^2\left(R_2r_n\right)}$$
$$\times\frac{1}{q\left[q+\alpha r_n^2q^{\beta}+\upsilon r_n^2\right]}. \quad (23)$$

We introduce the notation $$F(q)=\frac{1}{q\left[q+\alpha r_n^2 q^{\beta}+\upsilon r_n^2\right]},\tag{24}$$

and rewrite it in the equivalent form

$$F(q)=\frac{q^{-1-\beta}}{\left(q^{1-\beta}+\alpha r_n^2\right)+\upsilon r_n^2 q^{-\beta}}=\sum_{k=0}^{\infty}\left(-\upsilon r_n^2\right)^k\frac{q^{-1-\beta-k\beta}}{\left(q^{1-\beta}+\alpha r_n^2\right)^{k+1}}.\tag{25}$$

In order to determine the inverse Laplace transform of the function $\overline{v}(r,q)$ we will use the following formulae

$$L^{-1}\left\{\frac{1}{q^a}\right\}=\frac{t^{a-1}}{\Gamma(a)};\ a>0,$$

$$L^{-1}\left\{\frac{q^b}{\left(q^a-d\right)^c}\right\}=G_{a,b,c}(d,t)=\sum_{j=0}^{\infty}\frac{\Gamma(c+j)d^j}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma\left[(c+j)a-b\right]};$$
$$\operatorname{Re}(ac-b)>0.$$

So we find that the velocity field $v(r,t)$ has the following form

$$v(r,t)=\frac{\Omega_1R_1^2\left(R_2^2-r^2\right)+\Omega_2R_2^2\left(r^2-R_1^2\right)}{\left(R_2^2-R_1^2\right)r}t$$
$$-\pi\sum_{n=1}^{\infty}\frac{J_1\left(R_1r_n\right)\left[R_2\Omega_2J_1\left(R_1r_n\right)-R_1\Omega_1J_1\left(R_2r_n\right)\right]B_1\left(rr_n\right)}{J_1^2\left(R_1r_n\right)-J_1^2\left(R_2r_n\right)}$$
$$\times\sum_{k=0}^{\infty}\left(-\upsilon r_n^2\right)^k G_{1-\beta,-1-\beta-k\beta,k+1}\left(-\alpha r_n^2,t\right),\tag{26}$$

or equivalently, $$v(r,t)=\frac{\Omega_1R_1^2\left(R_2^2-r^2\right)+\Omega_2R_2^2\left(r^2-R_1^2\right)}{\left(R_2^2-R_1^2\right)r}t$$
$$-\pi\sum_{n=1}^{\infty}\frac{J_1\left(R_1r_n\right)\left[R_2\Omega_2J_1\left(R_1r_n\right)-R_1\Omega_1J_1\left(R_2r_n\right)\right]B_1\left(rr_n\right)}{J_1^2\left(R_1r_n\right)-J_1^2\left(R_2r_n\right)}$$

$$\times \sum_{j,k=0}^{\infty} \frac{\left(-\upsilon r_n^2\right)^k \left(-\alpha r_n^2\right)^j \Gamma\left(k+j+1\right)}{\Gamma\left(k+1\right)\Gamma\left(j+1\right)} \frac{t^{(1-\beta)j+k+1}}{\Gamma\left[\left(1-\beta\right)j+k+2\right]}. \tag{27}$$

2.3. Calculation of the Sheer Stress

The shear stress $\tau\left(r,t\right)$ is obtained from Eqs. (7) and (23). Applying the Laplace transform to Eq. (7), we find

$$\bar{\tau}\left(r,q\right)=\left(\mu+\alpha_1 q^{\beta}\right)\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{v}\left(r,q\right). \tag{28}$$

Now, differentiating Eq. (23) with respect to r and replacing the values of $\dfrac{\partial \bar{v}\left(r,q\right)}{\partial r}$ and $\bar{v}\left(r,q\right)$ into Eq. (28), we get

$$\bar{\tau}\left(r,q\right)=\frac{2R_1^2R_2^2\left(\Omega_2-\Omega_1\right)}{\left(R_2^2-R_1^2\right)r^2}\left(\mu\frac{1}{q^2}+\alpha_1\frac{1}{q^{2-\beta}}\right)+\pi\sum_{n=1}^{\infty}\left[\frac{1}{r}B_1\left(rr_n\right)-r_nB\left(rr_n\right)\right]$$

$$\times\frac{J_1\left(R_1r_n\right)\left[R_2\Omega_2J_1\left(R_1r_n\right)-R_1\Omega_1J_1\left(R_2r_n\right)\right]}{J_1^2\left(R_1r_n\right)-J_1^2\left(R_2r_n\right)}$$

$$\times\sum_{j,k=0}^{\infty}\frac{\left(-\upsilon r_n^2\right)^k\left(-\alpha r_n^2\right)^j\Gamma\left(k+j+1\right)}{\Gamma\left(k+1\right)\Gamma\left(j+1\right)}$$

$$\times\left[\mu\frac{1}{q^{(1-\beta)j+k+2}}+\alpha_1\frac{1}{q^{(1-\beta)j+k+2-\beta}}\right], \tag{29}$$

where $\quad B\left(rr_n\right)=J_0\left(rr_n\right)Y_1\left(R_2r_n\right)-J_1\left(R_2r_n\right)Y_0\left(rr_n\right).$

Applying inverse Laplace transform to the image function $\bar{\tau}\left(r,q\right)$, we find that the shear stress $\tau\left(r,t\right)$ under form

$$\tau\left(r,t\right)=\frac{2R_1^2R_2^2\left(\Omega_2-\Omega_1\right)}{\left(R_2^2-R_1^2\right)r^2}\left(\mu t+\frac{\alpha_1t^{1-\beta}}{\Gamma\left(2-\beta\right)}\right)+\pi\sum_{n=1}^{\infty}\left[\frac{1}{r}B_1\left(rr_n\right)-r_nB\left(rr_n\right)\right]$$

$$\times\frac{J_1(R_1r_n)\left[R_2\Omega_2J_1(R_1r_n)-R_1\Omega_1J_1(R_2r_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}$$

$$\times\sum_{j,k=0}^{\infty}\frac{\left(-\upsilon r_n^2\right)^k\left(-\alpha r_n^2\right)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}$$

$$\times\left[\mu\frac{t^{(1-\beta)j+k+1}}{\Gamma\left[(1-\beta)j+k+2\right]}+\alpha_1\frac{t^{(1-\beta)j+k+1-\beta}}{\Gamma\left[(1-\beta)j+k+2-\beta\right]}\right]. \quad (30)$$

3. LIMITING CASE

Making $\beta=1$ into Eqs. (26) and (30), we obtain the velocity field

$$v(r,t)=\frac{\Omega_1R_1^2\left(R_2^2-r^2\right)+\Omega_2R_2^2\left(r^2-R_1^2\right)}{\left(R_2^2-R_1^2\right)r}t$$

$$-\pi\sum_{n=1}^{\infty}\frac{J_1(R_1r_n)\left[R_2\Omega_2J_1(R_1r_n)-R_1\Omega_1J_1(R_2r_n)\right]B_1(R_1r_n)}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}$$

$$\times\sum_{k=0}^{\infty}\left(-\upsilon r_n^2\right)^kG_{0,-2-k,k+1}\left(-\alpha r_n^2,t\right), \quad (31)$$

and associated shear stress

$$\tau(r,t)=\frac{2R_1^2R_2^2\left(\Omega_2-\Omega_1\right)}{\left(R_2^2-R_1^2\right)r^2}\left(\mu t+\alpha_1\right)+\pi\sum_{n=1}^{\infty}\left[\frac{1}{r}B_1(rr_n)-r_nB(rr_n)\right]$$

$$\times\frac{J_1(R_1r_n)\left[R_2\Omega_2J_1(R_1r_n)-R_1\Omega_1J_1(R_2r_n)\right]}{J_1^2(R_1r_n)-J_1^2(R_2r_n)}$$

$$\times\sum_{j,k=0}^{\infty}\frac{\left(-\upsilon r_n^2\right)^k\left(-\alpha r_n^2\right)^j\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\left[\mu\frac{t^{k+1}}{\Gamma(k+2)}+\alpha_1\frac{t^k}{\Gamma(k+1)}\right], \quad (32)$$

corresponding to an ordinary second grade fluid, performing the same motion. The above relation can be simplified if we use the following relations:

$$\sum_{k=0}^{\infty}\left(-\upsilon r_n^2\right)^k G_{0,-2-k,k+1}\left(-\alpha r_n^2,t\right)=\sum_{k=0}^{\infty}\left(-\upsilon r_n^2\right)^k \sum_{j=0}^{\infty}\frac{\left(-\alpha r_n^2\right)^j \Gamma\left(k+j+1\right)}{\Gamma\left(k+1\right)\Gamma\left(j+1\right)}\frac{t^{k+1}}{\Gamma\left(k+2\right)}$$

$$=\sum_{k=0}^{\infty}\frac{\left(-\nu r_n^2\right)^k t^{k+1}}{\Gamma\left(k+2\right)}\frac{1}{\left(1+\alpha r_n^2\right)^{k+1}}$$

$$=-\frac{1}{\nu r_n^2}\sum_{k=0}^{\infty}\frac{1}{\left(k+1\right)!}\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)^{k+1}$$

$$=\frac{1}{\nu r_n^2}\left[1-\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right].$$

As a result, we find the velocity field (30) and the shear stress (32) under simplified forms

$$v\left(r,t\right)=\frac{\Omega_1 R_1^2\left(R_2^2-r^2\right)+\Omega_2 R_2^2\left(r^2-R_1^2\right)}{\left(R_2^2-R_1^2\right)r}t$$
$$-\frac{\pi}{\nu}\sum_{n=1}^{\infty}\frac{J_1\left(R_1 r_n\right)\left[R_2\Omega_2 J_1\left(R_1 r_n\right)-R_1\Omega_1 J_1\left(R_2 r_n\right)\right]}{J_1^2\left(R_1 r_n\right)-J_1^2\left(R_2 r_n\right)}$$
$$\times\frac{B_1\left(r r_n\right)}{r_n^2}\left[1-\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right], \quad (33)$$

$$\tau\left(r,t\right)=\frac{2R_1^2 R_2^2\left(\Omega_2-\Omega_1\right)}{\left(R_2^2-R_1^2\right)r^2}\left(\mu t+\alpha_1\right)+\pi\rho\sum_{n=1}^{\infty}\left[\frac{1}{r}B_1\left(r r_n\right)-r_n B\left(r r_n\right)\right]$$
$$\times\frac{J_1\left(R_1 r_n\right)\left[R_2\Omega_2 J_1\left(R_1 r_n\right)-R_1\Omega_1 J_1\left(R_2 r_n\right)\right]}{r_n^2\left[J_1^2\left(R_1 r_n\right)-J_1^2\left(R_2 r_n\right)\right]}$$
$$\times\left[1-\frac{1}{1+\alpha r_n^2}\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right]. \quad (34)$$

Eqs. (33) and (34) are identical with those obtained by Fetecau et al [8]. Making $\alpha=0$ into Eqs. (33) and (34), the similar solutions corresponding to the Newtonian fluid, performing the same motion, are recovered. Making $\Omega_1 = 0$ and $\Omega_2 = \Omega$ or $\Omega_1 = \Omega$ and $\Omega_2 = 0$ in any one of the above relations (26) and (30), we obtain the velocity field and the adequate shear stress corresponding to the flow between two cylinders, one of them being at rest. Finally, the similar solutions for Newtonian fluids are immediately obtained from Eqs. (33) and (34) for $\alpha \to 0$ (equivalently $\alpha_1 \to 0$).

4. CONCLUSION

In this paper, we have established exact solutions for the velocity field and the associated shear stress corresponding to the flow of a generalized second grade fluid between two infinite concentric circular cylinders. The motion is produced by the two cylinders which at time t = 0^+ begin to rotate around their common axis with angular velocities $\Omega_1 t$ and $\Omega_2 t$. The solutions, obtained by means of Laplace and Hankel transforms, are presented under integral and series forms in terms of the generalized $G_{a,b,c}(\cdot,\cdot)$ function, and satisfy all imposed initial and boundary conditions. For $\beta = 1$ or $\beta = 1$ and $\alpha = 0$, the similar solutions for the ordinary second grade fluids, respectively, Newtonian fluids are recovered. The velocity field and the adequate shear stress corresponding to the flow between two cylinders, one of them being at rest, are obtained as particular cases of our general solutions. Making $\Omega_1 = 0$ and $\Omega_2 = \Omega$ into Eq. (27), for instance, we obtain the velocity field

$$v(r,t) = \frac{\Omega R_2^2 \left(r^2 - R_1^2\right)}{\left(R_2^2 - R_1^2\right) r} t$$
$$-\pi R_2 \Omega \sum_{n=1}^{\infty} \frac{J_1^2(R_1 r_n) B_1(r r_n)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}$$
$$\times \sum_{j,k=0}^{\infty} \frac{\left(-\upsilon r_n^2\right)^k \left(-\alpha r_n^2\right)^j \Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)} \frac{t^{(1-\beta)j+k+1}}{\Gamma\left[(1-\beta)j+k+2\right]}, \tag{35}$$

corresponding to the flow between cylinders, the inner cylinder being at rest.

REFERENCES

[1] Bagley, R.L. and Torvik, P.J.; "A theoretical basis for the application of fractional calculus to viscoelasticity", J. Rheol., Vol. 27, pp. 201-210, 1983.

[2] Bagley, R.L. and Torvik, P.J.; "On the fractional calculus model of viscoelastic behavior", J. Rheol., Vol. 30, pp.133-155, 1986.

[3] Caputo, M. and Mainardi, F.; "A new dissipation model based on memory mechanism", Pure Appl. Geophys., Vol. 91, pp. 134-147, 1971.

[4] Debnath, L.; “Recent applications of fractional calculus to science and engineering”, Int. J. Math. and Math. Sci., Vol. 54, pp. 3413-3442, 2003.

[5] Dunn, J.E. and Rajagopal, K.R.; “Fluid of differential type: critical review and thermodynamics analysis”, Int. J. Eng. Sci., Vol. 33, pp. 689-728, 1995.

[6] Ferry, J.D., Landel, R.F. and William M.L.; “Extension of the Rouse theory of viscoelastic properties to undiluted linear polymers”, J. Appl. Phys., Vol. 26, pp. 359-362, 1955.

[7] Fetecau, C. and Fetecau C.; “Starting solutions for the motion of a second grade fluid due to longitudinal and torsional oscillations of a circular cylinder”, Int. J. Eng. Sci., Vol. 44, pp. 788-796, 2006.

[8] Fetecau, C., Fetecau C. and Vieru, D.; “On some helical flows of Oldroyd-B fluids”, Acta Mech., Vol. 189, pp. 53-63, 2007.

[9] Fetecau, C., Mahmood, A., Fetecau C. and Vieru, D.; “Some exact solutions for the helical flow of a generalized Oldroyd-B fluid in a circular cylinder”, Comp. & Appl. Math., Vol. 56, pp. 3096-3108, 2008.

[10] Hilfer, R.; “Applications of Fractional Calculus in Physics”, World Scientific Press, Singapore, 2000.

[11] Khan, M., Nadeem, S., Hayat, T. and Siddiqui, A.M.; “Unsteady motion of a generalized second grade fluid”, Math. Comput. Modell., Vol. 41, pp. 629-637, 2005.

[12] Koeller, R.C.; “Applications of fractional calculus to the theory of viscoelasticity”, Trans. ASME J. Appl. Mech., Vol. 51(2), pp. 299-307, 1984.

[13] Podlubny, I.; “Fractional Differential Equations”, Academic Press, San Diego, 1999.

[14] Rajagopal, K.R.; “A note on unsteady unidirectional flows of a non-Newtonian fluid”, Int. J. Non-Linear Mech., Vol. 17, pp. 369-371, 1982.

[15] Rogers, L.; “Operators and fractional derivatives for viscoelastic constitutive equations”, J. Rheol., Vol. 27, pp. 351-372, 1983.

[16] Rouse, P.E.; “The theory of the linear viscoelastic properties of dilute solutions of coiling polymers”, J. Chem. Phys., Vol. 21, pp. 1272-1280, 1953.

[17] Slonimsky, G.L.; “On the law of deformation of highly elastic polymeric bodies”, Dokl. Akad. Nauk, BSSR, Vol. 140, pp. 343-346, 1961.

[18] Stiassnie, M.; “On the application of fractional calculus for the formulation of viscoelastic models”, Appl. Math. Modelling, Vol. 3, pp. 300-302, 1979.

[19] Tan, W., Pan, W. and Xu, M.; “A note on unsteady flows of a viscoelastic fluid with the fractional Maxwell model between two parallel plates”, Int. J. Non-Linear Mech., Vol. 38, pp. 645-650, 2003.

[20] Tong, D. and Liu, Y.; “Exact solutions for the unsteady rotational flow of non-Newtonian fluid in an annular pipe”, Int. J. Eng. Sci., Vol. 43, pp. 281-289, 2005.

[21] Tong, D., Wand, R. and Yang, H.; “Exact solutions for the flow of non-Newtonian fluid with fractional derivative in an annular pipe”, Science in China Ser. G Physics, Mechanics & Astronomy, Vol. 48, pp. 485-495, 2005.

[22] Xu, M. and Tan, W.; “Theoretical analysis of the velocity field, stress field and vortex sheet of generalized second order fluid with fractional anomalous diffusion”, Science in China, Ser. A, Vol. 44(11), pp. 1387-1399, 2001.

[23] Xu, M. and Tan, W.; “The representation of the constitutive equation of viscielastic materials by the generalized fractional element networks and its generalized solutions”, Science in China, Ser. G, Vol. 46(2), pp. 145-157, 2003.